Gödel's Program and Two Dogmas of Set Theory
Talk 3: Inner Models and the Powerset Operation

The 2026 Tarski Lectures

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The inner model program

The Constructible Universe

Definition: Gödel (1940)

Suppose X is a set or a class.

- $L_0[X] = \emptyset$,

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Remark

- L is the $\subseteq$-minimal transitive model of ZFC that contains all ordinals.
- $L(X)$ is the $\subseteq$-minimal transitive model of ZF containing X and all ordinals.

Jensen's fine structural analysis of L

Fine structure

In *The fine structure of the constructible hierarchy* (1972) Ronald Jensen gave a complete level-by-level analysis (fine structural analysis) of L proving that $\diamondsuit$, $\square$ and many other principles hold in L.

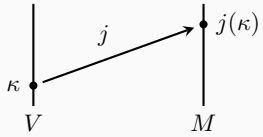
Two Definitions of Measurability

Definition: Measurability 1

Via Ultrafilters:
An uncountable cardinal κ is **measurable** if there exists a κ-complete, non-principal ultrafilter U on κ.

Definition: Measurability 2

Via Embeddings:



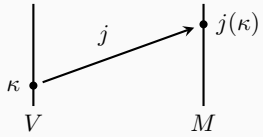
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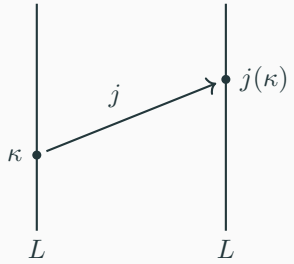
Theorem: Scott (1961)

Measurability 1 is equivalent to Measurability 2.
(An ultrafilter U yields an embedding via the ultrapower construction $V \to \text{Ult}(V, U)$, and an embedding j yields an ultrafilter via $X \in U \iff \kappa \in j(X)$).

Scott's Theorem

Theorem: Scott, 1961

Assume there is a measurable cardinal.
Then $V \neq L$.



j is the ultrapower embedding by the normal ultrafilter on κ.

The Inner Model Problem: the first attempt

Construct L-like (e.g. minimal, fully analyzable...) models for large cardinals

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Construct L-like (e.g. minimal, fully analyzable...) models for large cardinals
such as measurable cardinals, strong cardinals, Woodin cardinals, superstrong cardinals, supercompact cardinals ...

Constructions producing canonical L-like models

Kunen's solution for a measurable cardinal

Theorem: Kunen, 1970

Suppose κ is a measurable cardinal and μ is a normal κ-complete ultrafilter on κ. Then, letting $\nu = \mu \cap L[\mu]$,

- $L[\mu] \vDash$ "κ is a measurable cardinal",
- $L[\mu] \vDash$ "ν is the unique normal κ-complete ultrafilter on κ".

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Conclusion

- $L[\mu]$ is a canonical L-like model for one measurable cardinal.
- $L[\mu]$ is the $\subseteq$-minimal model of ZFC that contains all ordinals and thinks that κ is a measurable cardinal.

The main idea

- Large cardinals are given via embeddings $j : V \to M$.

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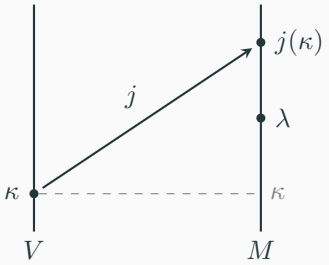
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- Large cardinals are given via embeddings $j : V \to M$.
- To construct L-like models that have the desired large cardinals, we need to construct a model K such that $j \restriction K \in K$.
- So, our model should somehow be of the form $L[j]$.
- Or $L[(j_\alpha : \alpha \in \mathrm{Ord})]$, as it should have many large cardinals.

Embeddings as extenders



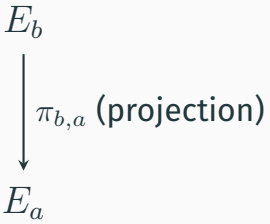
Definition: (κ, λ)-Extender derived from j

The (κ, λ)-extender derived from j is the set E consisting of pairs (a, A) such that

- $a \in [\lambda]^{<\omega}$,
- $A \in \mathscr{P}([\kappa]^{|a|})$,
- $a \in j(A)$.

Extenders as sequence of measures

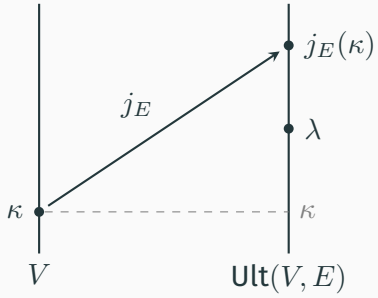
- Extenders can be thought of as a coherent sequence of ultrafilters indexed by $[\lambda]^{<\omega}$.



If $a \subseteq b$, the measure E_b projects to E_a.

Extenders as sequence of measures

- Extenders can be thought of as a coherent sequence of ultrafilters indexed by $[\lambda]^{<\omega}$.
- We can think of j as the ultrapower embedding by an extender.



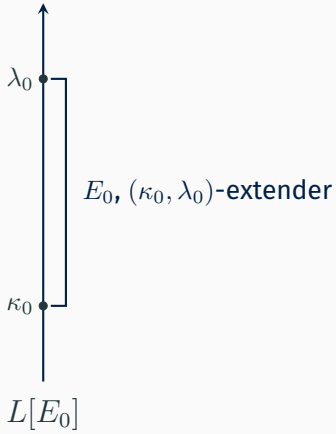
$j_E \colon V \to \mathsf{Ult}(V, E)$

K^c constructions: building $L[\vec{E}]$

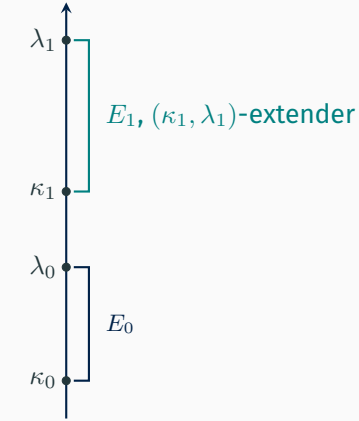


L

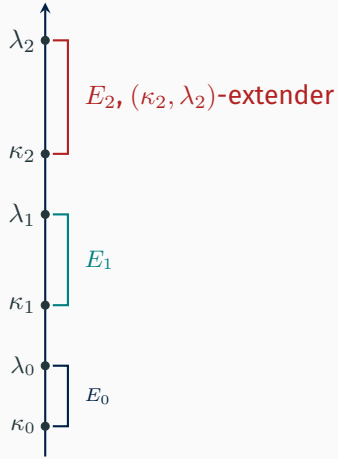
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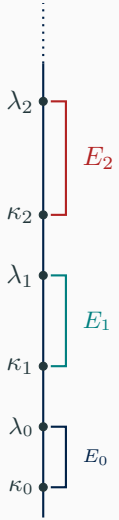


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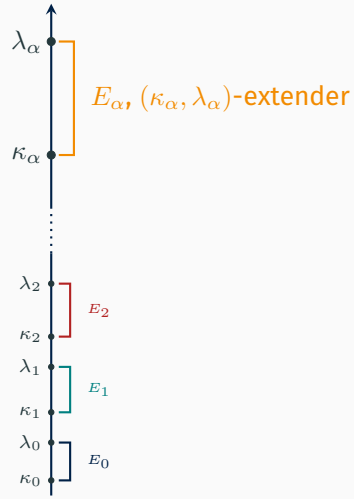
$L[E_0, E_1, E_2]$

K^c constructions: building $L[\vec{E}]$



$L[E_0, E_1, E_2, \dots]$

K^c constructions: building $L[\vec{E}]$



$L[E_0, \ldots, E_\alpha]$

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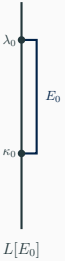


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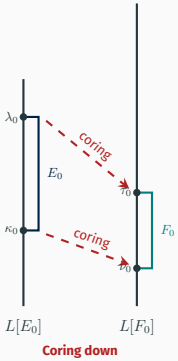
If only things were so easy.

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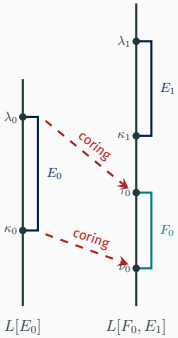


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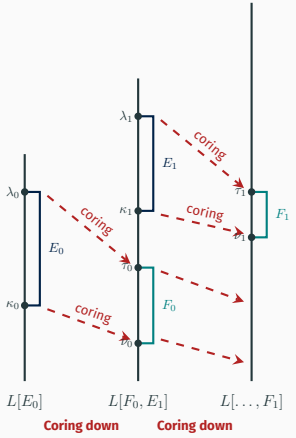
Coring down

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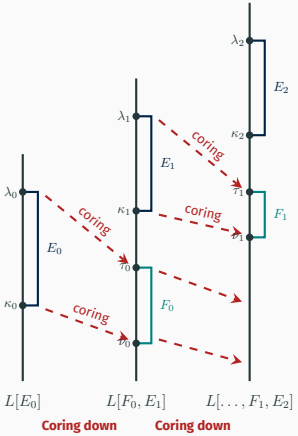


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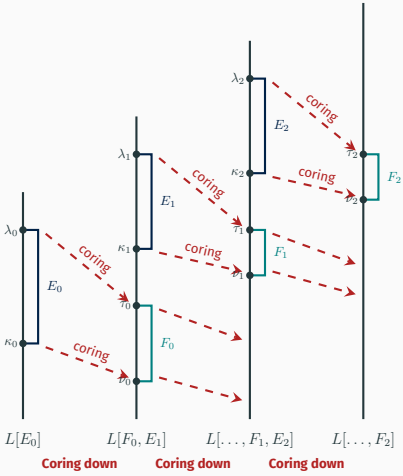
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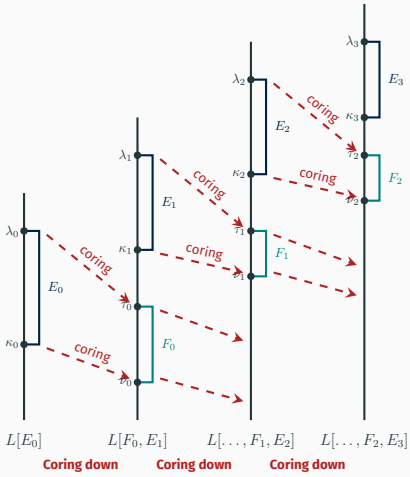
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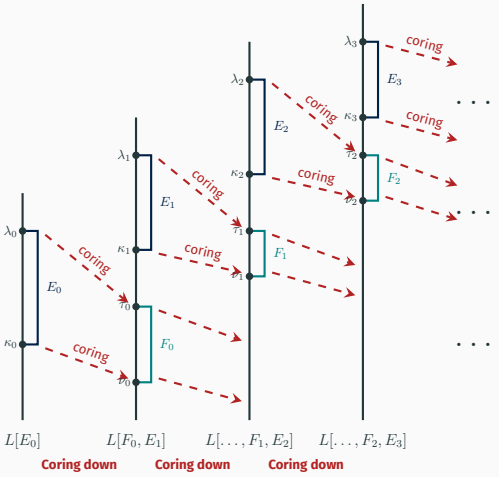
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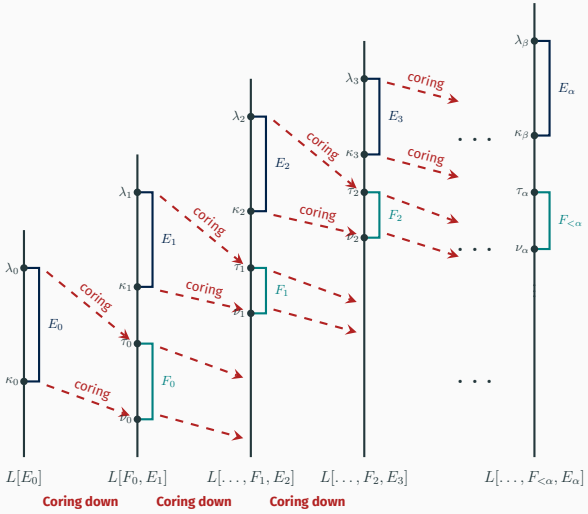
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- The process may or may not converge to a model of ZFC.
- Jensen, Mitchell, Schindler, Steel, and others showed that various K^c constructions converge modulo an iterability conjecture.
- Iterability conjectures state that countable hulls of models appearing in the K^c construction are $(\omega_1 + 1)$-iterable.

Iteration trees and iterability conjectures

Linear Iteration via a Normal Measure

$$M_0$$

Linear Iteration via a Normal Measure

$$M_0 \xrightarrow{j_0} M_1 = \mathrm{Ult}(M_0, \mu_0)$$

- $\mu_0 \in M_0$ is a κ_0-complete normal ultrafilter on κ_0.
- κ_i and μ_i are the images of κ_0 and μ_0.

Linear Iteration via a Normal Measure

$$M_0 \xrightarrow{j_0} \begin{matrix} M_1 \\ = \mathrm{Ult}(M_0, \mu_0) \end{matrix} \xrightarrow{j_1} \begin{matrix} M_2 \\ = \mathrm{Ult}(M_1, \mu_1) \end{matrix}$$

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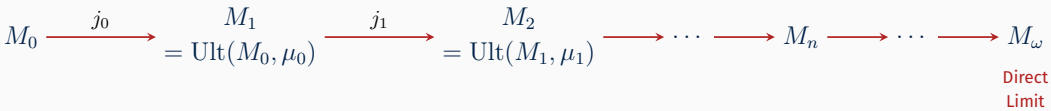
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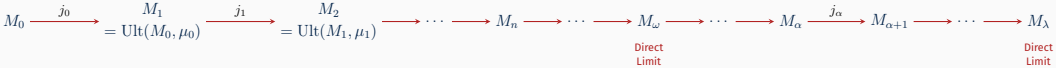
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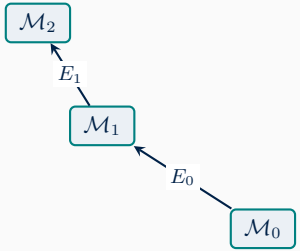


- $\mu_0 \in M_0$ is a κ_0-complete normal ultrafilter on κ_0.
- κ_i and μ_i are the images of κ_0 and μ_0.
- M_0 **is iterable if each** M_α **is well-founded.**

Iteration Trees ($\mathcal{T}$)

The Structure of $\mathcal{T}$

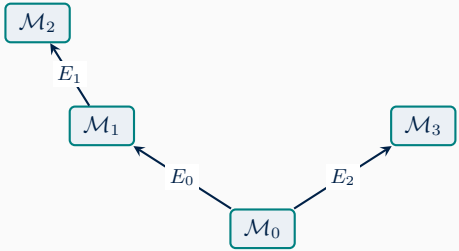
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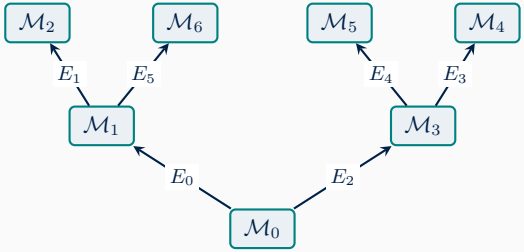
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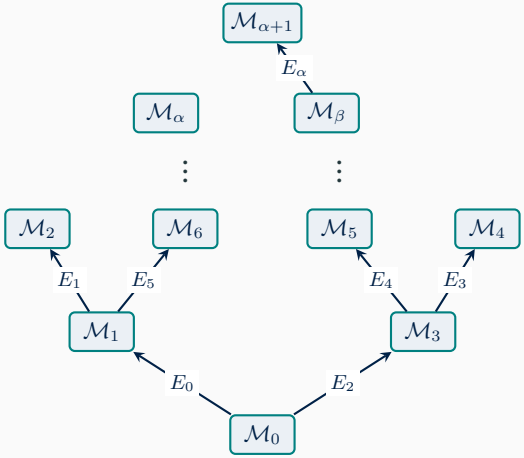
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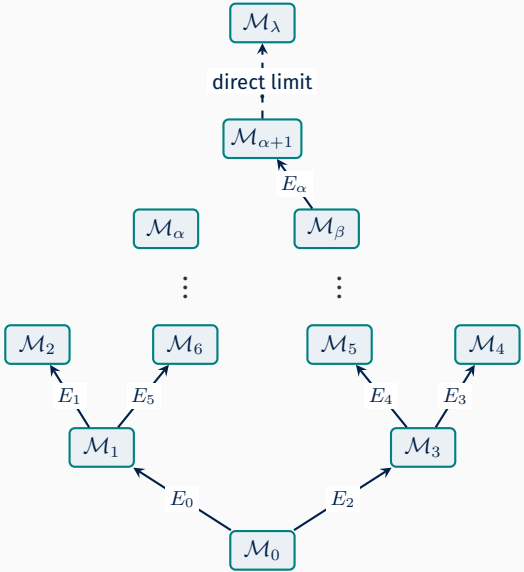
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- This creates $\mathcal{M}_{\alpha+1}$ and a new embedding $\pi_{\beta,\alpha} : \mathcal{M}_\beta \to \mathcal{M}_{\alpha+1}$.



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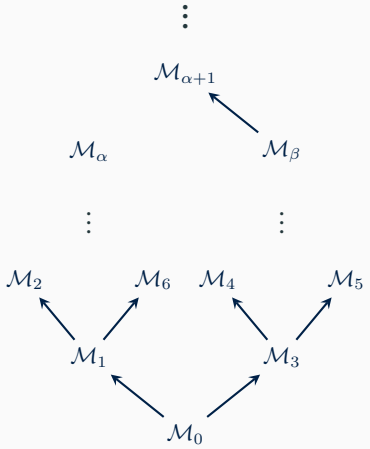
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- This creates $\mathcal{M}_{\alpha+1}$ and a new embedding $\pi_{\beta,\alpha} : \mathcal{M}_\beta \to \mathcal{M}_{\alpha+1}$.
- Limit models $\mathcal{M}_\lambda$ are formed via direct limits along a branch.



Iteration trees

Remark

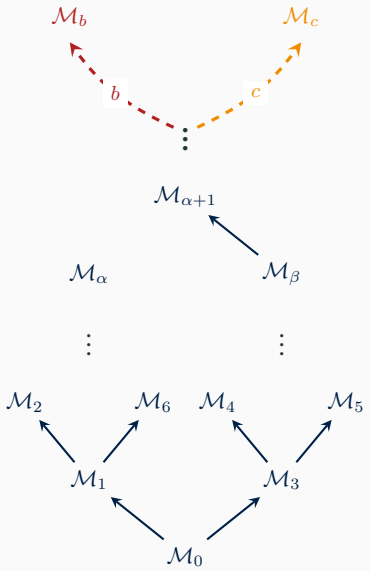
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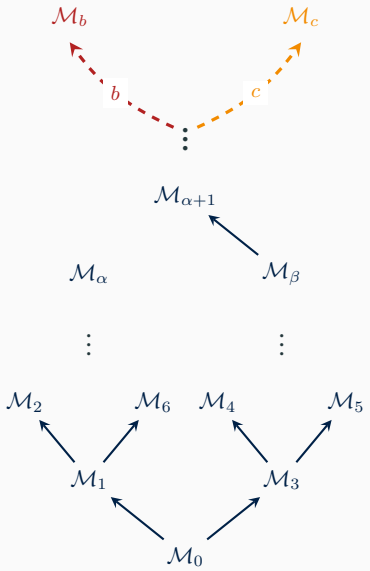
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Iteration trees

Remark

- We need to pick a branch at limit steps.
- There maybe many or none at all.
- We want to make sure all models are well-founded, but we may not be able to pick a branch that achieves this.



Branch of an iteration tree



Iterability

Definition:

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Definition:

We say M is $(\omega_1 + 1)$-iterable if there is a function Σ as above that assigns branches to all iteration trees of length $\leq \omega_1 + 1$ that have root M.

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Recall

Iterability conjectures state that countable hulls of models appearing in the K^c construction are $(\omega_1 + 1)$-iterable.

The inner model problem

Definition:

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The inner model problem

Construct mice with large cardinals.

A Timeline of Inner Model Theory



Gödel (1938): Constructs L, the universe of constructible sets.

A Timeline of Inner Model Theory



Scott (1961): Proves there are no measurable cardinals in L.

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Jensen, Schimmerling, Schindler, Steel (2009): Powerful covering theorem for K^c.

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The Achilles' heel of K^c

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If N is a minimal Nairian model, then the Mitchell-Schindler K^c construction, along with many other versions of K^c, fails in a generic extension of N.

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Survival of the fittest

Many other versions of K^c have survived, in particular, fully backgrounded K^c models.

Descriptive inner model theory

Trees with no branches

Aronszajn trees

ZFC proves that there are trees of height ω_1 with no branches.

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Remark

- The construction of iteration strategies that pick branches at stage ω_1 is hard.
- We need to make sure our strategy is definable.
- So we need to bring in our definability context, AD and uB (the universally Baire sets).

Hod mice rushing to save mice

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- A **hod mouse** is an $(\omega_1 + 1)$-iterable model of the form $\mathcal{M} = L[\vec{E}, \Sigma]$ where Σ is the strategy of $\mathcal{M}$.

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Turns out HODs of models of AD^+ are hod mice.

A Timeline of HOD Mice and HOD Calculations



Steel (1995): Seminal work showing $\mathrm{HOD}^{L(\mathbb{R})}$ up to θ is a mouse.

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A Timeline of HOD Mice and HOD Calculations

Steel (2016–2020s): Develops the general theory of hod mice; shows under HPC that HOD of models of $\mathrm{AD}_{\mathbb{R}}$ are hod mice, and gets a Karp Prize.

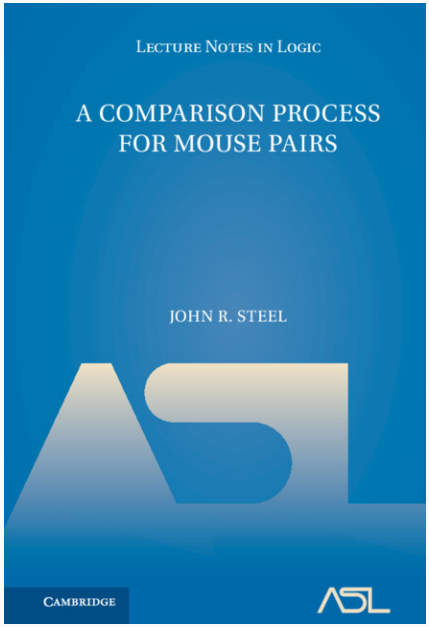
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LECTURE NOTES IN LOGIC
A COMPARISON PROCESS FOR MOUSE PAIRS
JOHN R. STEEL
ASL
CAMBRIDGE
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The inner model problem

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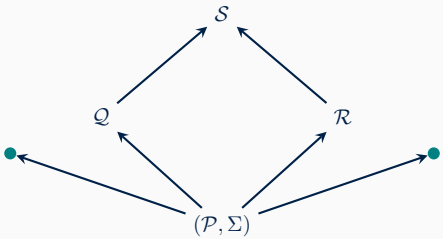
- Show that HOD's of models of AD^+ have large cardinals.
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- Show that MM implies HOD of $L(\mathrm{uB}, \mathbb{R})$ has large cardinals.

HOD is a hod mouse

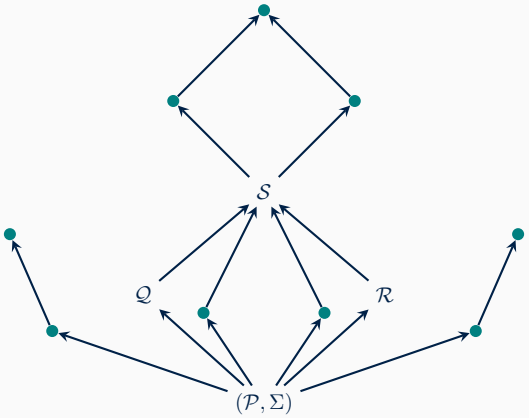
Directed systems generated by hod pairs

$(\mathcal{P}, \Sigma)$

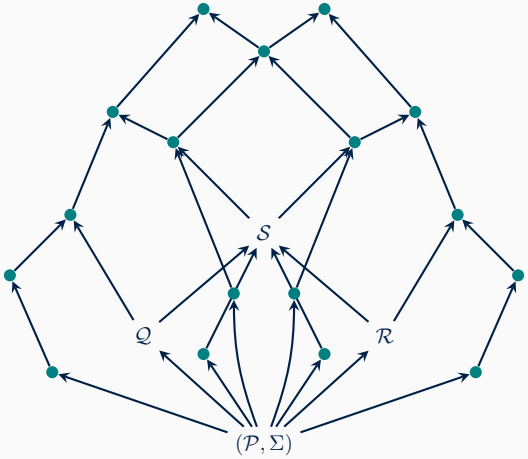
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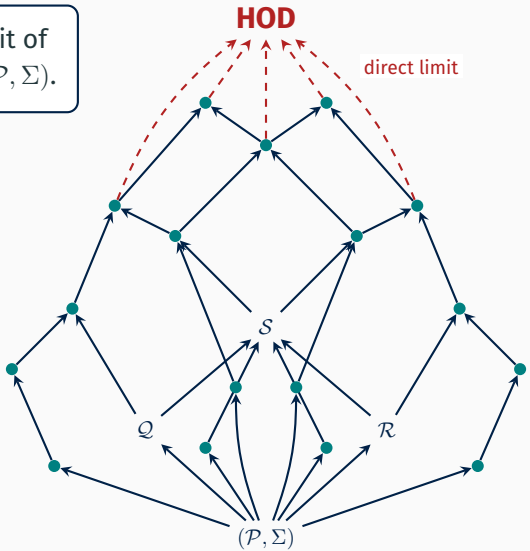


Directed systems generated by hod pairs



Directed systems generated by hod pairs

HOD is the directed limit of countable iterates of $(\mathcal{P}, \Sigma)$.



HPC

Recursion theory and set theory

- (Kleene and others) HYP $= \Delta^1_1 = \mathbb{R} \cap L_{\omega_1^{\mathsf{CK}}}$.

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Definition: Steel, 2020

Assume ZF $+$ $\mathsf{AD}_{\mathbb{R}}$. HPC is the statement that for every set of reals A, there is a hod pair $(\mathcal{P}, \Sigma)$ such that A is definable over the structure $(\mathsf{HC}, \Sigma, \in)$.

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Theorem: S. 2020

Assume ZF $+$ AD$_{\mathbb{R}}$ and that there is no hod mouse with a **Woodin cardinal that is a limit of Woodin cardinals**. Then HPC holds.

From HPC to HOD analysis

Theorem: Steel, 2020

Assume ZF $+$ $\mathsf{AD}_{\mathbb{R}}$ $+$ HPC $+$ $V = L(\mathscr{P}(\mathbb{R}))$. Then HOD is a hod premouse.

$L(\mathrm{uB}, \mathbb{R})$

uB sets

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- uB is the set of universally Baire sets.
- Recall that these are those sets of reals that can be extended to generic extensions in a meaning preserving way.
- If A is uB and g is generic, then A_g is the interpretation of A in $V[g]$.
- $\mathrm{uB}_g = (\mathrm{uB})^{V[g]}$ and $\mathbb{R}_g = \mathbb{R}^{V[g]}$.

Towards Sealing

Theorem: Woodin, 1990s

Assume there is a proper class of Woodin cardinals and suppose A is a universally Baire set. Then for any V-generic g, there is an elementary embedding

$$j : L(A, \mathbb{R}) \to L(A_g, \mathbb{R}_g)$$

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Question

Is there such a generic absoluteness theorem for $L(\mathrm{uB}, \mathbb{R})$?

Definition: Woodin, 2004

Sealing is the following statement.

- There is a class of Woodin cardinals.
- $L(\mathrm{uB}, \mathbb{R}) \vDash \mathsf{AD}^+$.
- For any V-generic g and $V[g]$-generic h, there is an embedding

$$j : L(\mathrm{uB}_g, \mathbb{R}_g) \to L(\mathrm{uB}_{g*h}, \mathbb{R}_{g*h})$$

such that $j(A) = A_h$.

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Suppose there is a class of Woodin cardinals and κ is a supercompact cardinal. let $g \subseteq \mathrm{Coll}(\omega, 2^\kappa)$ be V-generic. Then $V[g] \vDash$ Sealing.

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A Remark

All canonical models (e.g. mice or hod mice) satisfy $L(\mathrm{uB}, \mathbb{R}) \vDash \neg$AD.

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All canonical models (e.g. mice or hod mice) satisfy $L(\mathrm{uB}, \mathbb{R}) \vDash \neg$AD.

Conjecture

Large cardinals do not imply that $L(\mathrm{uB}, \mathbb{R}) \vDash$ AD.

The universe below Woodin limit of Woodin cardinals.

Sealing **is weak**

Theorem: S., Trang, 2018

The consistency strength of Sealing is weaker than a **Woodin cardinal that is a limit of Woodin cardinals**.

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Theorem: Müller, S., Siskind, 2026

Suppose $\mathcal{M}$ is the minimal mouse with a Woodin cardinal that is a limit of Woodin cardinals. Let δ be this Woodin cardinal and let κ be the least $< \delta$-strong cardinal that reflects the set of strong cardinals. Let $g \subseteq \mathrm{Coll}(\omega, 2^\kappa)$ be generic. Then

$$\mathcal{M}|\delta[g] \vDash \text{Sealing}.$$

Powerset is Σ^1_2 in L

Suppose $\kappa \leq \lambda$ are cardinals of L and $g \subseteq \mathrm{Coll}(\omega, \lambda)$ is generic. Then

$$\mathscr{P}(\kappa) \cap L$$

is Σ^1_2 definable in $L[g]$ from κ.

Powerset in mice

Suppose

- $V = \mathcal{M}$ is the minimal mouse with a Woodin cardinal that is a limit of Woodin cardinals,
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- It is not definable in $L(\mathrm{uB}_g, \mathbb{R}_g)$ from $\mathcal{M}|\kappa$.
- Otherwise, we could define an ω_1-sequence of reals in $L(\mathrm{uB}_g, \mathbb{R}_g)$ contradicting Sealing.

The universe below WLW

- We tend to think that the axiom

  ZFC+“there is a Woodin cardinal that is a limit of Woodin cardinals”

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Theorem:

The consistency strengths of the following statements are not stronger than a **Woodin cardinal that is a limit of Woodin cardinals**.

- (S., 2009) ZF $+ \Theta_{\text{reg}}$.
- (S., 2023) ZF + LSA.
- (S., 2009) ZFC + MM(c).
- (S., 2009) ZFC + CH + "there is an ω_1-dense ideal on ω_1".
- (S., Trang, 2023) Sealing.
- (Blue, Larson, S., 2025) ZFC $+ \forall \kappa \geq \omega_1 (\neg \square_\kappa)$.
- (S., 2025) ZFC + MM(c) + SRP(ω_3).

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- (Mohammadpour, S. 2025-26) ZFC + GCH + "ω_1 is generically supercompact" (i.e., there are dense ideals at every cardinal).
- (Blue, Larson, S., 2025) Failure of HOD Hypothesis.
- (Blue, Larson, S., 2024) Failure of the Iterability Conjecture for the Mitchell-Schindler K^c.
- (Woodin, 2000s) The existence of divergent models of AD^+.
- (Neeman, 96-2000s) Determinacy of various long games.
- ...

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- In fact, huge amount of structure exists below it.
- Moreover, below this cardinal, the powerset operation is beyond any definability notion currently known to descriptive set theorists.

IMP for WLW

Theorem: Neeman, 2002

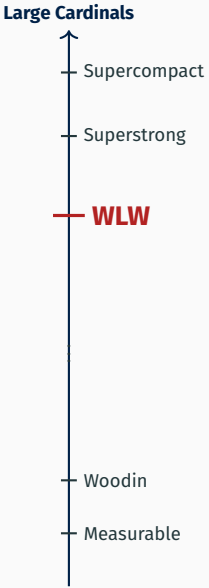
Assume there is a class of Woodin cardinals that are limits of Woodin cardinals. Then there is a mouse with a Woodin cardinal that is a limit of Woodin cardinals.

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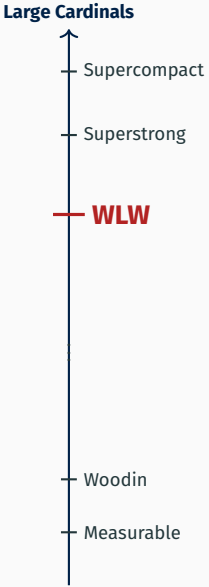
- While the above is the best result on IMP, it covers a huge chunk of the universe, not a small chunk (as is often said).



Beyond wlw

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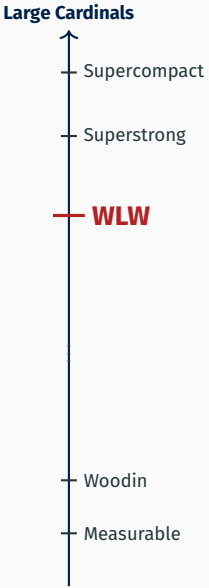
- descriptive set theoretic hierarchies currently unknown to us,



Beyond wlw

To go beyond, we need to discover

- descriptive set theoretic hierarchies currently unknown to us,
- verifiable traces of large cardinals in structures like H_{ω_3} (i.e. interesting combinatorial, descriptive set theoretic or other kinds of properties).



Extending uB sets

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- Basically, we need to develop a theory of universally Baire subsets of uB sets, call it uB_2. Then we need to get uB_3, and so on.

A philosophical challenge

> Can one argue that AD holds in $L(\mathrm{uB}, \mathbb{R})$?


Thank You
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