Gödel's Program and Two Dogmas of Set Theory
Talk 2: Nairian Models

The 2026 Tarski Lectures

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Nairian Model $= L(\mathrm{HOD}, \mathrm{Ord}^{\omega})$

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- In Armenian culture, Նաիրի (Nairi) is the ancient name of Armenia, remembered in literature and poetry as a majestic, peaceful, and poetic realm to which we must return.
- What unifies Determinacy with Forcing Axioms must be majestic, and hence the name.

Quine’s Web of Belief

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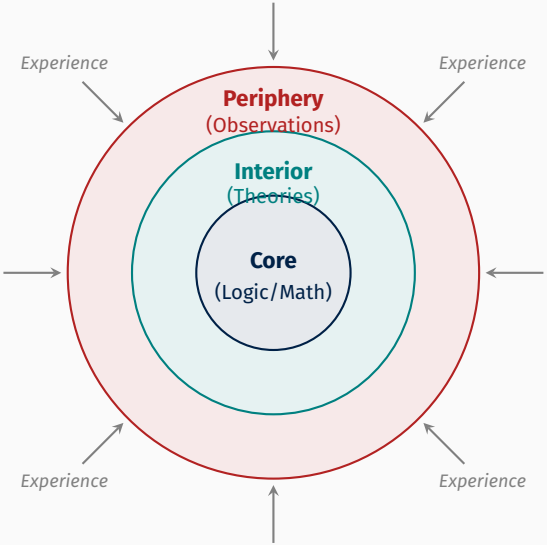
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- **Interior:** E.g., scientific theories.
- **Periphery:** Beliefs we can easily give up. E.g., observation sentences.



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- When a belief resists revision, it gradually moves to the center.

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- T is consistent if and only if some large cardinal axiom implies $\mathrm{Con}(T)$.
- Large cardinals predict the consistency strengths of set-theoretic statements correctly.

Large cardinals correctly predict consistency strengths

A core large-cardinal centric belief

- Suppose ψ is some (natural) set-theoretic statement (e.g. ψ is AD, MM, $\forall \kappa \geq \omega_1(\neg \square_\kappa)$).

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- Suppose ψ is some (natural) set-theoretic statement (e.g. ψ is AD, MM, $\forall \kappa \geq \omega_1(\neg \square_\kappa)$).
- If ϕ is the weakest large cardinal such that $\mathrm{Con}(\phi)$ implies $\mathrm{Con}(\psi)$ in a natural way, then $\mathrm{Con}(\psi)$ implies $\mathrm{Con}(\phi)$.

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- If ϕ is the weakest large cardinal such that $\mathrm{Con}(\phi)$ implies $\mathrm{Con}(\psi)$ in a natural way, then $\mathrm{Con}(\psi)$ implies $\mathrm{Con}(\phi)$.
- I.e., ϕ and ψ are equiconsistent.

Examples: MM, Failure of square principles, SRP, Θ_{reg}, Dense Ideals

Jensen's Square Principle ($\square_\kappa$)

Definition: Jensen (1972)

For an uncountable cardinal κ, the principle $\square_\kappa$ asserts the existence of a sequence

$$\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ is a limit ordinal} \rangle$$

satisfying the following three conditions:

1. **Club:** C_α is a closed unbounded (club) subset of α.
2. **Boundedness:** The order type of C_α is at most κ (i.e., $\mathrm{otp}(C_\alpha) \leq \kappa$).
3. **Coherence:** If β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.
4. (Key Point) C_{κ^+} cannot exist.

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Strength

- $\square_\kappa$ is one of the most important set-theoretic principles that played a prominent role in Jensen's study of L.
- All prior known ways of forcing the failure of square everywhere required a supercompact cardinal.
- Some individual instances of the failure of square at singular cardinals have been obtained by Martin Zeman from weaker hypotheses (though still stronger than a superstrong cardinal).

Holy Grails of inner model theory

Conjecture

Each of the following statements implies the existence of a (canonical) model with a **superstrong cardinal**:

- ZFC + "There is a supercompact cardinal",
- ZFC + "There is a strongly compact cardinal",
- ZFC + MM,
- ZFC + $\forall \kappa \geq \omega_1 (\neg \square_\kappa)$,
- ZFC + SRP (not in the original list),
- ZFC + CH + "there is an ω_1-dense ideal on ω_1",
- ZF + Θ_{reg}.

— John Steel, *Derived models associated to mice* (2007)
— John Steel, PFA *implies* $\text{AD}^{L(\mathbb{R})}$ (2005)

The consistency strength of some of the Holy Grails

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The theory

ZFC+“there is a Woodin cardinal that is a limit of Woodin cardinals”

occupies a much lower place in the hierarchy of large cardinal axioms than **superstrong cardinals**



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occupies a much lower place in the hierarchy of large cardinal axioms than **superstrong cardinals** which themselves are placed much lower than **supercompact cardinals**.



The strength of Holy Grails

Theorem:

The consistency strengths of the following statements are not stronger than a **Woodin cardinal that is a limit of Woodin cardinals**.

- (S., 2009) ZF $+ \Theta_{\text{reg}}$.
- (S., 2023) ZF + LSA.
- (S., 2009) ZFC + MM(c).
- (S., 2009) ZFC + CH + "there is an ω_1-dense ideal on ω_1".
- (S., Trang, 2023) Sealing.
- (Blue, Larson, S., 2025) ZFC $+ \forall \kappa \geq \omega_1 (\neg \square_\kappa)$.
- (S., 2025) ZFC + MM(c) + SRP(ω_3).

The strength of Holy Grails, continued

Theorem:

The consistency strengths of the following statements are not stronger than a **Woodin cardinal that is a limit of Woodin cardinals**.

- (Mohammadpour, S. 2025-26) ZFC + GCH + "ω_1 is generically supercompact" (i.e., there are dense ideals at every cardinal).
- (Blue, Larson, S., 2025) Failure of HOD Hypothesis.
- (Blue, Larson, S., 2024) Failure of the Iterability Conjecture for the Mitchell-Schindler K^c.
- (Woodin, 2000s) The existence of divergent models of AD^+.
- (Neeman, 96-2000s) Determinacy of various long games.

Conjecture

- The Strong Reflection Principle is weaker than a Woodin cardinal that is a limit of Woodin cardinals.

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- MM can be forced over Nairian Models.

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- Forcing Moore's **Mapping Reflection Principle** or Viale-Weiss's **guessing principles** or even MM for **Namba forcing** might be much harder to force over Nairian Models, or maybe even impossible.
- We may need to have a better understanding of MM − SRP.

A revision

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> Science is not absolute certainty. We must rely on our current best theories to test and improve them, and we must be open to revisions.

What has really happened

- Prior to Nairian Models, all known models of AD had a highly non-compact structure above $\mathscr{P}(\mathbb{R})$ as they had the form $L(\mathscr{P}(\mathbb{R}))$.

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- Prior to Nairian Models, all known models of AD had a highly non-compact structure above $\mathscr{P}(\mathbb{R})$ as they had the form $L(\mathscr{P}(\mathbb{R}))$.
- Compactness is highly desirable and is a sign of the universe being saturated with many interesting mathematical structures.
- Nairian Models are compact everywhere (not just in the portion coded in the reals).

The Large-Cardinal-Centric view of V

The Large-Cardinal Regime

- Start with a V that has large cardinals.

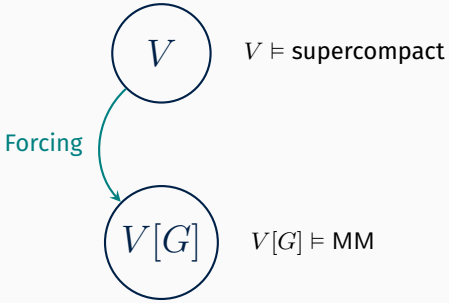


$V \vDash$ supercompact

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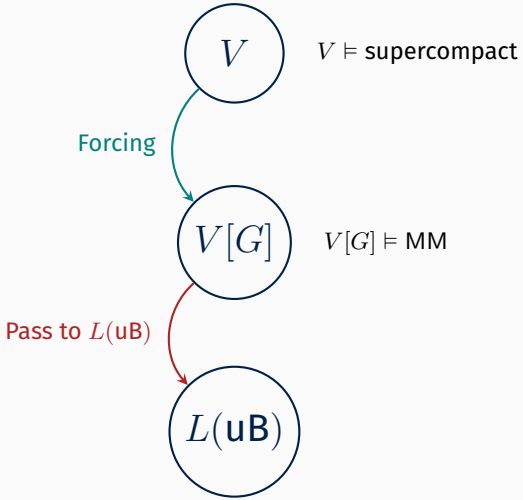
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- Force to pass to our favorite framework.



The Large-Cardinal-Centric view of V

The Large-Cardinal Regime

- Start with a V that has large cardinals.
- Force to pass to our favorite framework.
- Then pass to $L(\mathsf{uB})$ (a model of AD).



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- The Nairian Models perspective gives an AD-centric view of the universe.

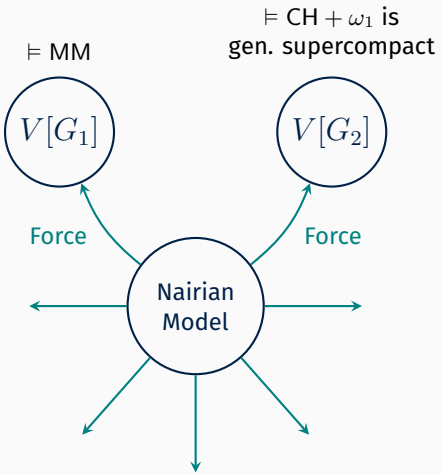
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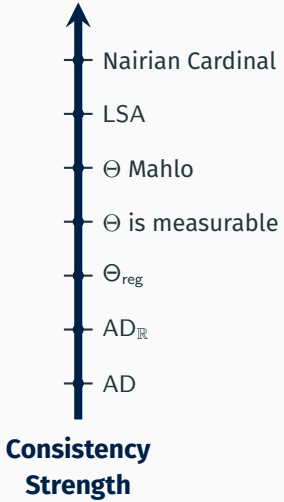
- Start with a model of AD that is compact everywhere (e.g., a Nairian Model).
- Force the desired ZFC framework you want to work in.



Forcing over models of AD

The Hierarchy of Determinacy Axioms

- AD.
- $\mathrm{AD}_{\mathbb{R}}$.
- $\Theta_{\mathrm{reg}} = \mathrm{AD}_{\mathbb{R}} + \Theta$ is regular.
- $\mathrm{AD}_{\mathbb{R}} + \Theta$ is a measurable cardinal.
- $\mathrm{AD}_{\mathbb{R}} + \Theta$ is a Mahlo cardinal.
- LSA (The Largest Suslin Axiom).
- Θ is a Nairian Cardinal.
- ...



History of forcing over models of AD



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The CLSSSZ result motivated the research into Nairian Models.

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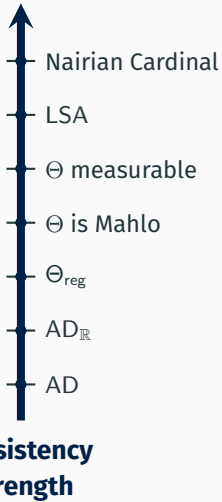
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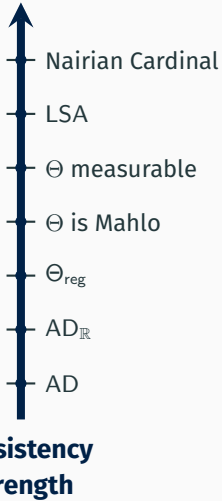
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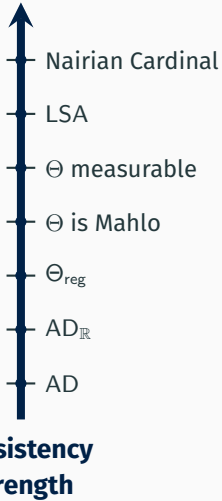
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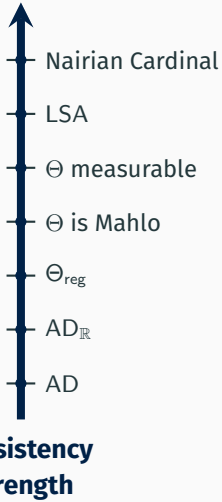
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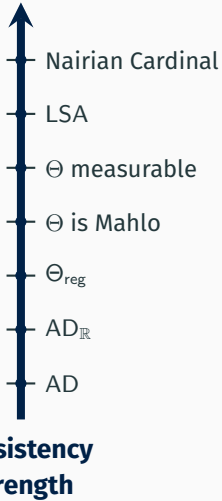
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- Sometimes it is the other way around. (E.g. the saturation of the non-stationary ideal on ω_1.).
- The Key Point: We should develop forcing over models of determinacy further.



The internal structure of the Nairian Models

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- Nairian Models are not of the form "L of a set".
- No reason why they should be of any interest.

Nairian Models are interesting

Theorem: Steel and Woodin, 2020-2025

Assume $V = L(\mathscr{P}(\mathbb{R})) + \text{LSA} + \text{HPC}$. Suppose Θ is a limit of Woodin cardinals in HOD and let N be the Nairian Model at Θ. Then $N \vDash \text{ZF}$.

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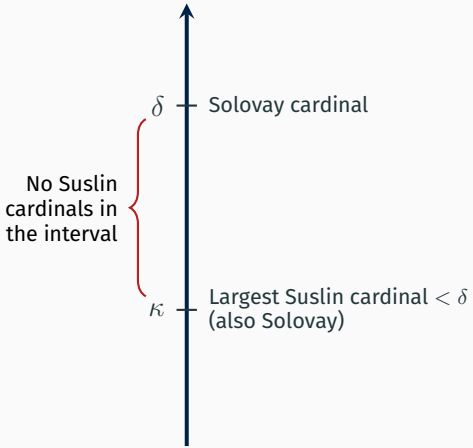
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- κ is a **Suslin cardinal** if there is a κ-Suslin A that is not λ-Suslin for any $\lambda < \kappa$.

Some key definitions

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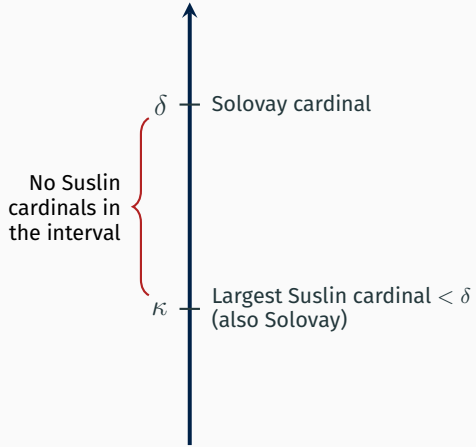
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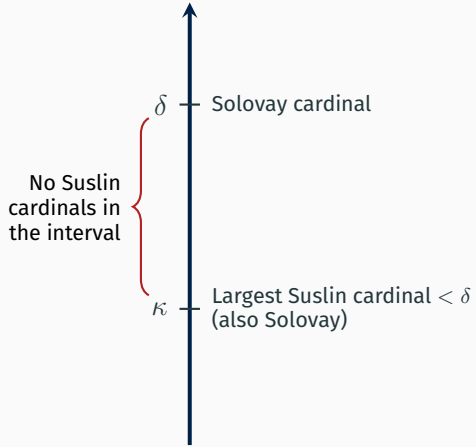
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- LSA (The Largest Suslin Axiom) is the theory ZF $+$ "Θ is an LSA cardinal".



Some key definitions

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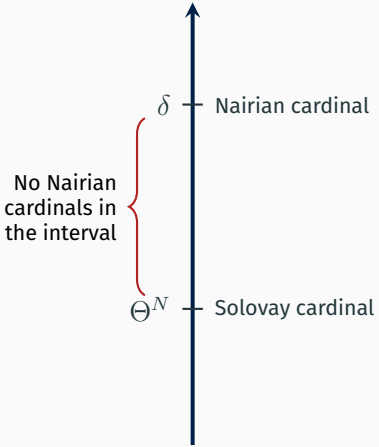
- δ is an LSA **cardinal** if δ is a Solovay cardinal and the largest Suslin cardinal $< \delta$ is also a Solovay cardinal.
- LSA (The Largest Suslin Axiom) is the theory ZF + "Θ is an LSA cardinal".
- LSA plays an important role in Woodin's $V = \text{Ultimate} - \text{L}$ Axiom.



Nairian cardinals

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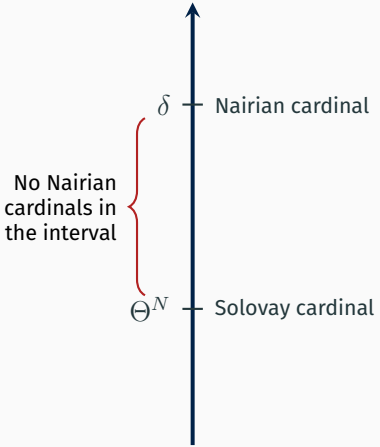
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Nairian cardinals

Definition:

- δ is a **Nairian cardinal** if the Nairian Model N at δ is a model of ZF and Θ^N is a Solovay cardinal.
- N is a **minimal Nairian model** if $\delta = \mathrm{Ord} \cap N$ is a Nairian cardinal and there are no Nairian cardinals in the interval (Θ^N, δ).



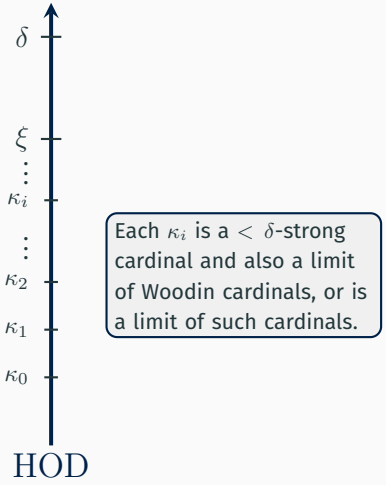
Minimal Nairian Models

Theorem: Larson, Blue, S., 2024

Assume $\Theta_{\text{reg}} + \text{HPC}$ and suppose δ is an LSA cardinal such that there is $\xi < \delta$ with the property that in HOD,

- ξ is inaccessible in HOD,
- ξ is a limit of Woodin cardinals, and
- ξ is a limit of $< \delta$-strong cardinals.

Then the least such ξ is a minimal Nairian cardinal, i.e., the Nairian Model at ξ is a minimal Nairian Model.



Minimal Nairian Models exist

Definition:

- We say N is a **minimal Nairian model** if it is a minimal Nairian model inside some model M of $\Theta_{\text{reg}} + \text{HPC}$.
- We say (N, M) is a **minimal Nairian pair** if N is a minimal Nairian model of M.

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Theorem: S. 2018-2024

Assume there is a Woodin cardinal that is a limit of Woodin cardinals. Then there is a minimal Nairian Model.

Definition:

If X is a set, then Θ_X is the supremum of all ordinals that are surjective images of X.

Supercompactness measures

Definition:

Assume ZF and suppose X is an infinite set. We say ω_1 is **X-supercompact** if there is a countably complete ultrafilter μ on $\mathscr{P}_{\omega_1}(X)$ (the countable subsets of X) such that

- (Fineness) For each $a \in X$, $\{Y \in \mathscr{P}_{\omega_1}(X) : a \in Y\} \in \mu$.

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- (Fineness) For each $a \in X$, $\{Y \in \mathscr{P}_{\omega_1}(X) : a \in Y\} \in \mu$.
- (Normality) For each $F : \mathscr{P}_{\omega_1}(X) \to X$, if for every $Y \in \mathscr{P}_{\omega_1}(X)$, $F(Y) \subseteq Y$, then there is $a \in X$ such that $\{Y \in \mathscr{P}_{\omega_1}(X) : a \in F(Y)\} \in \mu$.

Supercompactness

Definition:

Assume ZF. We say ω_1 is **supercompact** if it is X-supercompact for every uncountable X.

Internal Structure of Nairian Models

Theorem:

Suppose (N, M) is a minimal Nairian pair. Set $\xi = \mathrm{Ord} \cap N$ and let $(\kappa_i : i < \xi)$ be the increasing enumeration of the strong cardinals of HOD^N and their limits. Then the following holds.

- $N \vDash \mathrm{ZF}$.

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- In M, $N^{<\kappa_0} \subseteq N$.
- If $g \subseteq \mathbb{P}_{\mathrm{max}}$ is M-generic, then in $M[g]$, $N[g]$ is closed under ω_1-sequences.

Conjecture

The following theories are equiconsistent:

- ZF $+$ “there is a supercompact cardinal”.
- ZF $+$ “ω_1 is a supercompact cardinal”.
- ZFC $+ \forall \kappa \geq \omega_1 (\neg \square_\kappa)$.

Towards the Strong Reflection Principle.

Strong Reflection Principle, SRP

SRP **is almost** MM

Many useful consequences of MM are already consequences of SRP (e.g., the saturation of NS, failure of square everywhere, ...).

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Definition:

We say $S \subseteq \mathscr{P}_{\omega_1}(\kappa)$ is stationary if for every $\alpha > \kappa$ and for every $F \subseteq V_\alpha$, there is a countable $X \prec (V_\alpha, F, \in)$ such that $X \cap \kappa \in S$.

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Definition:

We say S as above is projective-stationary if for every stationary $T \subseteq \omega_1$, the set $\{X \in S : X \cap \omega_1 \in T\}$ is stationary.

Strong Reflection Principle, SRP

Definition: SRP**, Feng-Jech 1994, Todorčević 1980s**

SRP(κ) is the statement that every projective stationary subset $S \subseteq \mathscr{P}_{\omega_1}(\kappa)$ contains a $\subseteq$-continuous chain of length ω_1.

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MM implies SRP.

Conjecture

SRP can be forced over any Minimal Nairian Model.

The quasi-club conjecture

Theorem:

Assume $\mathsf{AD}_{\mathbb{R}}$ and X is a surjective image of $\mathbb{R}$.

- (Solovay, 1978) Then the club filter is an ultrafilter on $\mathscr{P}_{\omega_1}(\mathbb{R})$.

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- (Woodin, 1983) There is a unique normal, fine countably complete ultrafilter on $\mathscr{P}_{\omega_1}(X)$, in fact, it is the quasi-club filter.

Conjecture

Suppose N is a minimal Nairian Model. Then for any uncountable α the quasi-club filter is an ultrafilter over $\mathscr{P}_{\omega_1}(\alpha^{\omega})$.

SRP **from the conjecture**

If the conjecture holds, then we can force SRP over N.

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Theorem: S., 2025

SRP(ω_3) can be forced over N.

A Nairian Theory

A hierarchical set

Definition:

A set M is **hierarchical** if M is transitive and, letting $\kappa = \mathrm{Ord} \cap M$, there exists a $\subseteq$-increasing continuous sequence $\langle M_\alpha : \alpha < \kappa \rangle$ of transitive sets such that

- $M = \bigcup_{\alpha<\kappa} M_\alpha$ and,
- for each $\alpha < \kappa$, $M_\alpha \in M$.

The $\mathbb{N}$-hierarchy

Definition:

H_X is the collection of sets of hereditarily size X.

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Definition:

The **N-hierarchy** $\langle \mathsf{N}_\alpha : \alpha \in \mathrm{Ord} \rangle$ is defined as follows:

- N_0 is the set of hereditarily finite sets.
- $\mathsf{N}_1 = H_{\mathsf{N}_0}$.
- For every α, $\mathsf{N}_{\alpha+2} = H_{\mathsf{N}_{\alpha+1}}$.
- For every limit ordinal α, $\mathsf{N}_\alpha = \bigcup_{\beta<\alpha} \mathsf{N}_\beta$, and $\mathsf{N}_{\alpha+1} = H_{\mathscr{P}(\mathsf{N}_\alpha)}$.

The Axiom of Harmony

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H-like

- Think of the usual properties of H_κ, the collection of sets of hereditarily size κ.
- E.g., if $f : \kappa \to \kappa^+$ then $\mathrm{rge}(f) \in H_\kappa$.

Nairian Theory

Definition:

The theory $\mathsf{NT}_{\mathsf{base}}$ consists of the following statements:

- $\mathsf{ZF} + \mathsf{AH}$.
- For every successor ordinal $\alpha \geq 2$, if X is of maximal cardinality in N_α, then $\mathsf{ref}(X, \mathsf{N}_\alpha, \mathsf{N}_{\alpha+1})$ holds.

The reflection principle $\mathrm{ref}(X, \mathsf{N}_\alpha, \mathsf{N}_{\alpha+1})$

Definition:

For sets X, Y and Z, $\mathrm{ref}(X, Y, Z)$ is the statement that whenever W is a transitive X-closed structure with $Y \in W$, and a is an element of W, there exists a transitive set M and an elementary embedding $\pi \colon M \to W$ such that

- $\{X, Y\} \in M$,

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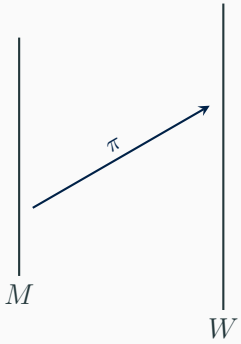
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- M is X-closed.

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$a \in \mathrm{rge}(\pi)$

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M is closed under X-sequences

$\mathsf{NT}_{\mathsf{base}}$ in Nairian Models

Theorem: Blue, Larson, S., 2026

Suppose N is a minimal Nairian model. Then $N \vDash \mathsf{NT}_{\mathsf{base}}$.

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Remark

We need $\mathsf{NT}_{\mathsf{base}}$ to force AC.

Forcing AC over (minimal) Nairian Models

Definition: The forcing

Let $\mathbb{Q}$ be the poset that adds a Cohen subset to every regular cardinal $\geq |\mathbb{R}|$ organized as a reverse Easton iteration.

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Theorem: Blue, Larson, S., 2026

Suppose $N \vDash \mathsf{AD}_{\mathbb{R}} + \mathsf{NT}_{\mathsf{base}}$ and suppose $G \subseteq \mathbb{P}_{\mathsf{max}} * \mathbb{Q}$ or $G \subseteq \mathrm{Coll}(\omega_1, \mathbb{R}) * \mathbb{Q}$ is N-generic. Then

$$N[G] \vDash \mathsf{AC}.$$

A conjecture

Conjecture

Suppose N is a transitive model of

$$\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \mathsf{NT}_{\mathsf{base}} + \text{“}\omega_1 \text{ is supercompact”}$$

and N has no inaccessible cardinals. Then $\mathbb{P}_{\mathsf{max}} * \mathbb{Q}$ forces

$$\mathsf{ZFC} + \mathsf{MM}(c) + \forall \kappa \geq \omega_1 (\neg \square_\kappa).$$

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Perhaps more needs to be added, e.g., $V = L(\mathrm{HOD}, \mathrm{Ord}^\omega)$.

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Suspicions

HOD Hypothesis

Definition: HOD Hypothesis, Woodin (2010)

There is a class of regular cardinals that are not (ω-strongly) measurable in HOD.

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HOD Conjecture, Woodin (2010)

Suppose there is an extendible cardinal. Then the HOD Hypothesis holds.

HOD Hypothesis over Nairian Models

Theorem: Ben-Neria, Hayut (2019)

It is consistent relative to a very small large cardinal that for any infinite regular κ, κ^+ is ω-strongly measurable in HOD.

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Theorem: Blue, Larson, S., 2025-2026

Suppose N is a minimal Nairian model. Then $\mathbb{P}_{\mathsf{max}} * \mathbb{Q}$ forces that the HOD Hypothesis fails. Thus, if $G \subseteq \mathbb{P}_{\mathsf{max}} * \mathbb{Q}$, then in $N[G]$, all uncountable regular cardinals are (ω-strongly) measurable in HOD.

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Remark

There is a tension between Nairian Models and the HOD Conjecture. The extendible cardinal is missing.

Woodin's perspective

- ZFC extensions of Nairian Models cannot have extendible cardinals.

Woodin's perspective

- ZFC extensions of Nairian Models cannot have extendible cardinals.
- In fact, they cannot have Woodin cardinals.

My perspective

Conjecture

- Assume there is no hod mouse with a strong cardinal that is also a Woodin cardinal. Then ZFC extensions of Nairian models do not have Woodin cardinals.

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- Assume there is no hod mouse with a strong cardinal that is also a Woodin cardinal. Then ZFC extensions of Nairian models do not have Woodin cardinals.
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- The key point is that requiring something as small as a Woodin boosts the strength of the underlying large cardinal hypothesis.

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Extendible cardinals, hallucinations

- The key point is that requiring something as small as a Woodin boosts the strength of the underlying large cardinal hypothesis.
- No traditional ZFC cardinal implies that there is a Nairian model whose ZFC extension has an extendible cardinal.
- The ZF cardinals (Reinhardt cardinals, ...) will imply the existence of Nairian Models whose ZFC forcing extensions have extendible cardinals.

The CH extensions

Models of CH from determinacy

Theorem: Mohammadpour, S., 2025-2026

Suppose N is a minimal Nairian model and suppose $G \subseteq \mathrm{Coll}(\omega_1, \mathbb{R}) * \mathbb{Q}$ is generic over N. Then the following statements hold in $N[G]$:

- For every $\kappa \geq \omega_1$, if $g \subseteq \mathrm{Coll}(\omega, \kappa)$ is generic, then in $N[G][g]$, there is an elementary embedding
$$j : N[G] \to M$$
such that $\mathrm{crit}(j) = \omega_1^{N[G]}$ and $M^\omega \subseteq M$.

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Suppose N is a minimal Nairian model and suppose $G \subseteq \mathrm{Coll}(\omega_1, \mathbb{R}) * \mathbb{Q}$ is generic over N. Then the following statements hold in $N[G]$:

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$$j : N[G] \to M$$
such that $\mathrm{crit}(j) = \omega_1^{N[G]}$ and $M^\omega \subseteq M$.
- For every $\kappa \geq \omega_1$, $\neg\square_\kappa$.

Larger Nairian Models

HOD-Berkeley cardinals

Definition: HOD-Berkeley cardinal

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- for every $\alpha > \kappa$,

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there is an embedding $j : V_\alpha^{\mathrm{HOD}} \to V_\alpha^{\mathrm{HOD}}$ such that

$$j(M) = M \text{ and } \mathrm{crit}(j) \in (\beta, \kappa).$$

HOD-Berkeley and HOD Conjecture

Theorem: Bagaria, Koellner, Woodin (2019)

Suppose there is an extendible cardinal and a HOD-Berkeley cardinal above it. Then the HOD Conjecture fails.

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- Are HOD-Berkeley cardinals consistent?

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Theorem: Bagaria, Koellner, Woodin (2019)

Suppose there is an extendible cardinal and a HOD-Berkeley cardinal above it. Then the HOD Conjecture fails.

Question

- Are HOD-Berkeley cardinals consistent?
- Is the theory

  $\mathrm{ZFC} +$ "there is an extendible cardinal and a class of HOD-Berkeley cardinals"

  consistent?

HOD-Berkeley cardinals from AD

- In 2024-2025, Blue and S. showed that under AD, ω_1 is $< \Theta$-Berkeley.

HOD-Berkeley cardinals from AD

- In 2024-2025, Blue and S. showed that under AD, ω_1 is $< \Theta$-Berkeley.
- It is natural to ask if ω_1 is HOD-Berkeley in some Nairian Model.

More Nairian Models

Definition:

- Suppose $\mathfrak{p} = (\mathcal{P}, \Sigma)$ is a hod pair. Let $\kappa_{\mathfrak{p}}$ be the least measurable cardinal of $\mathcal{P}$, and let $\mathcal{M}_\infty(\mathfrak{p}, \kappa_{\mathfrak{p}})$ be the direct limit of all iterates of $\mathfrak{p}$ that are above $\kappa_{\mathfrak{p}}$.

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- Let $N^+|\lambda = L_\lambda(\mathcal{F}|\lambda, \cup_{\beta<\lambda}\beta^\omega)$.

Theorem: S.

Assume $\mathsf{AD}_{\mathbb{R}} + \Theta_{\mathsf{reg}} + \mathsf{HPC}$. Suppose (ξ, δ) are such that δ is a Solovay cardinal and ξ is the least inaccessible cardinal of HOD that is a limit of Woodin cardinals and $< \delta$-strong cardinals. Then

$$N^+|\xi \vDash \mathsf{ZF} + \text{“}\omega_1 \text{ is } \mathrm{HOD}\text{-Berkeley”}.$$

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Conjecture

If (ξ, δ) are as above and $G \subseteq \mathbb{P}_{\mathsf{max}} * \mathbb{Q}$ is generic over $N^+|\xi$, then

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Disclaimer

The proof is not fully verified.

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Can any of the following be forced over a model of determinacy?

- Rado's Conjecture.
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- Interesting combinatorial principles that are not consequences of MM.


Thank You
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