Gödel's Program and Two Dogmas of Set Theory
Talk 1: Gödel's Program

The 2026 Tarski Lectures

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Summary

- Gödel's Program
- Nairian Models
- Inner Models and the powerset operation.

The main message is

to develop the method of forcing over models of determinacy in parallel with the method of forcing over models of large cardinals.

The Continuum Hypothesis

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Zermelo-Fraenkel Set Theory

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- Basic rules that govern our usage of sets.
- E.g., Union, Pairing, Replacement, Powerset...
- And ...the Axiom of Choice (AC).
- Thus, ZF and ZFC $=$ ZF $+$ AC.

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While cardinality is the most basic set-theoretic concept, neither ZF nor ZFC provides any tools to decide whether two arbitrary (yet concrete) infinite sets have the same cardinality.

The Continuum Hypothesis (CH)

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- Every infinite set of real numbers is either countable or has the same cardinality as the reals.
- Formally, if $A \subseteq \mathbb{R}$ is infinite, then

$$|A| = \aleph_0 \text{ or } |A| = |\mathbb{R}|.$$

CH after AC, 1890s

The Continuum Hypothesis, Cantor, Version 2, 1890s

- $|\mathbb{R}| = \aleph_1$.
- $2^{\aleph_0} = \aleph_1$.

The Independence of CH and Gödel's Program

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The Independence of CH

- **Gödel (1940):** $\mathrm{Con}(\mathrm{ZF}) \implies \mathrm{Con}(\mathrm{ZFC} + \mathrm{CH})$ (via the constructible universe L).
- **Cohen (1963):** $\mathrm{Con}(\mathrm{ZFC}) \implies \mathrm{Con}(\mathrm{ZFC} + \neg\mathrm{CH})$ (via the method of forcing).

Gödel's Program

"For if the meanings of the primitive terms of set theory [...] are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality."

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Gödel's Program

- Remove undecidability from set theory/mathematics in natural extensions of ZFC.
- Remove undecidability from set theory/mathematics in large cardinal extensions (axioms of higher infinity) of ZFC.

Gödel's Program

"There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory."

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Gödel's Program, Steel's Formulation

Decide mathematically interesting questions independent of ZFC in well-justified extensions of ZFC.

— John Steel, *Gödel's Program* (2014)

Feferman on Gödel's Program

- ...Gödel made a variety of suggestions, but most frequently he emphasized the route of introducing ever higher axioms of infinity.
- Despite the intense exploration of the "higher infinite" in the last 30-odd years, no single rationale of that character has emerged. Moreover, CH still remains undecided by such axioms, though they have been demonstrated to have many other interesting set-theoretical consequences.

— Solomon Feferman, *Gödel's program for new axioms: Why, where, how and what?* (2017)

The Levy-Solovay Phenomenon (1967)

Large cardinal axioms do not decide the Continuum Hypothesis.

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Large cardinal axioms do not decide the Continuum Hypothesis.
Thus, the LC-centric interpretation of Gödel's program fails for CH.

The Large Cardinal Hierarchy

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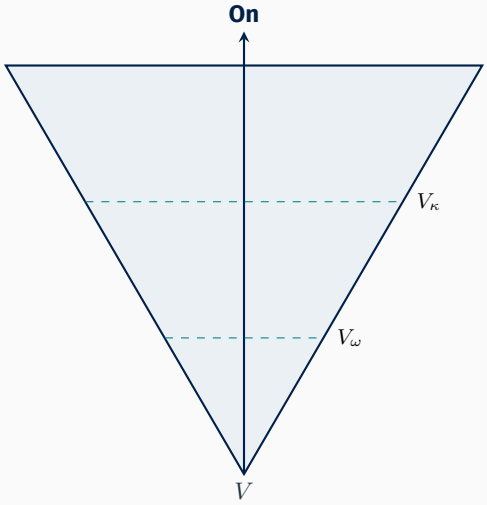
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- Large cardinal axioms are reflection axioms.
- κ is a large cardinal if V_κ "looks like" V.

The wrong picture of V



The right picture of V



Large cardinal axioms as critical points of elementary embeddings

Definition: Elementary embeddings

- Suppose M and N are transitive models or class models of some fragment of ZFC. We say $j : M \to N$ is an **elementary embedding** if for any formula (or property) ϕ and for any finite tuple $(a_1, \ldots, a_n) \in M^{<\omega}$,

$$M \vDash \phi[a_1, \ldots, a_n] \iff N \vDash \phi[j(a_1), \ldots, j(a_n)]$$

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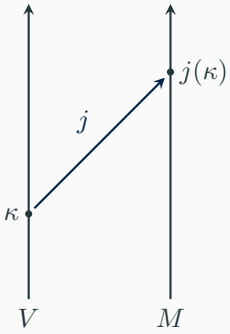
Large cardinals

We say κ is a **large cardinal** (as witnessed by an elementary embedding) if there is a transitive class M and an elementary embedding $j : V \to M$ such that

$$j \neq \mathrm{id} \text{ and } \mathrm{crit}(j) = \kappa.$$

Large Cardinals

We get stronger and stronger LC-axioms by requiring that M is close to V.

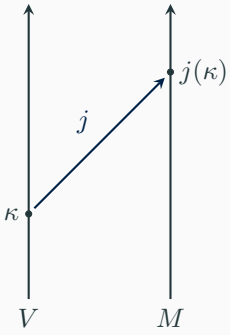


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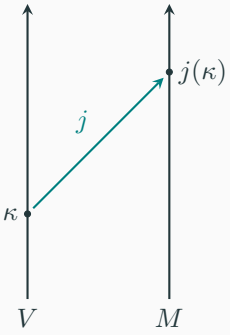
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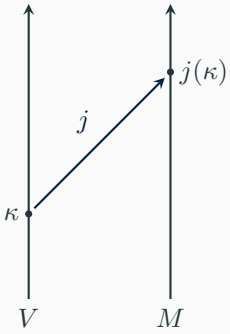


$V_{j(\kappa)} \subseteq M$

κ is Superstrong

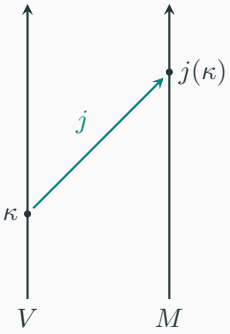
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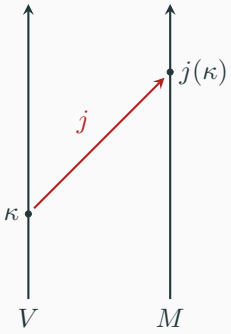
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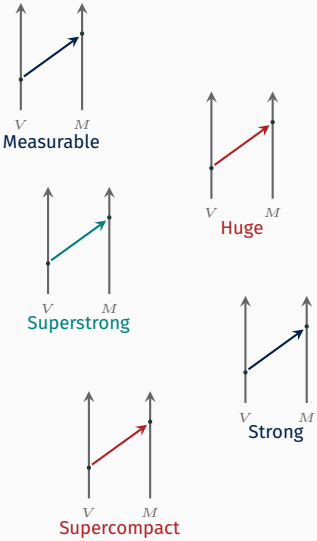


$M^{j(\kappa)} \subseteq M$

κ is Huge

Large Cardinals

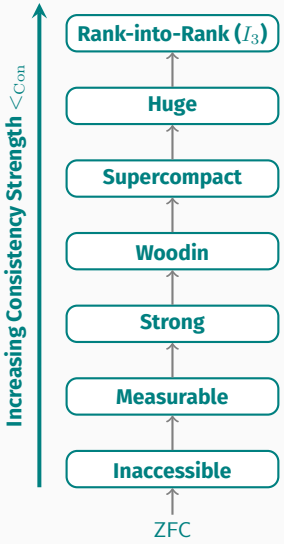
- Inaccessible cardinals
- Weakly compact cardinals
- Ramsey Cardinals
- Measurable cardinals
- Strong cardinals
- Woodin cardinals
- A Woodin cardinal that is a limit of Woodin Cardinals
- Shelah cardinals
- Superstrong cardinals
- Subcompact cardinals
- Supercompact cardinals
- Extendible Cardinal
- Huge cardinals
- I_0 cardinals
- ...



The consistency strength hierarchy and Large Cardinals

The Consistency Strength Hierarchy

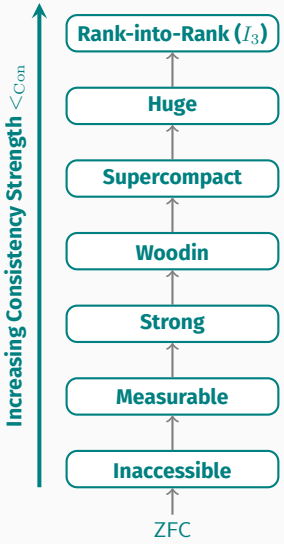
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- Thus, $S <_{\mathrm{Con}} T$ if T proves that S has a model.



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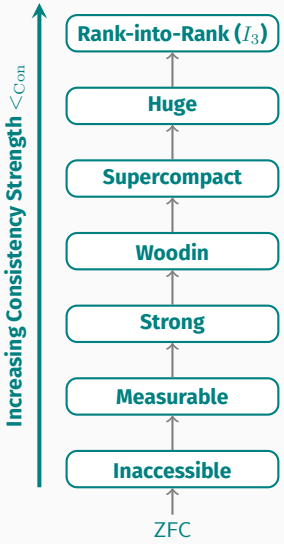
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The Large Cardinal Hierarchy: A Phenomenon

- Large cardinals form a consistency strength hierarchy.
- This means that $<_{\mathrm{Con}}$ more or less linearly orders the large cardinal axioms.



The Large-Cardinal-Centric view of V

The Large-Cardinal Regime

- Large cardinal axioms are consistent.

The Large-Cardinal-Centric view of V

The Large-Cardinal Regime

- Large cardinal axioms are consistent.
- T is consistent if and only if some large cardinal axiom proves $\mathrm{Con}(T)$.

Independence proofs: The constructible universe

Building new models from old ones: Inner Models

Theorem: Gödel (1940)

$\mathrm{Con}(\mathrm{ZFC}) \implies \mathrm{Con}(\mathrm{ZFC} + \mathrm{CH})$. In fact, GCH holds in L.

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Theorem: Jensen, (1972)

Suppose $\kappa \geq \omega_1$ is such that $X \in V_\kappa$. Then for every $\lambda \geq \kappa$, $\square_\lambda$ holds in $L(X)$.

Independence proofs: The method of forcing

Building new models from old ones: Forcing

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- The construction of $V[g]$ is analogous to the construction of field extensions.
- To show that $V[g] \vDash$ ZFC, we need to close $V[g]$ under much more complicated operations.

Soul searching

Summary

The two versions of CH

- **(Version 1)** Suppose A is an infinite subset of the reals. Then $|A| = |\mathbb{N}|$ or $|A| = |\mathbb{R}|$.
- **(Version 2)** $|\mathbb{R}| = \aleph_1$ (or $2^{\aleph_0} = \aleph_1$).

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- What do mathematicians usually do?
- Look for structures, examples—just do something!

Towards Version 1 of CH.

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- (The Baire Space) $\mathbb{R} = \omega^\omega$ and the basic open sets are

$$N_s = \{x \in \omega^\omega : s \trianglelefteq x\}$$

where $s \in \omega^{<\omega}$.

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- **Projective sets:** Sets obtained from Borel sets by repeatedly taking projections and complements (finitely many times).
- Sets in $L(\mathbb{R})$.
- ...
- Universally Baire sets.

The Universally Baire Sets

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Definition: Feng, Magidor, and Woodin (1992)

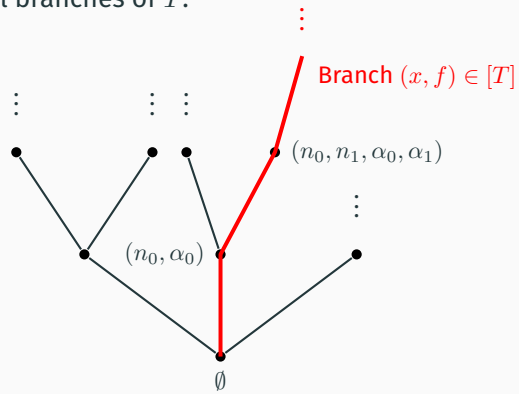
$A \subseteq \mathbb{R}$ is **universally Baire** if its continuous preimages in all compact Hausdorff topological spaces have the property of Baire.

The Universally Baire Sets: a better definition

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Definition: Feng, Magidor, Woodin, (1992)

$A \subseteq \mathbb{R}$ is **universally Baire** if for every κ there are trees T and S on $\omega \times \kappa$ such that

- $p[T] = A$.
- for any generic g for a poset of size $< \kappa$,

$$V[g] \vDash p[T] = \mathbb{R} - p[S]$$

Interpretations of uB sets

Definition:

- Suppose A is a uB set, g is generic and (T, S) witnesses that A is uB. Then

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Theorem: Woodin, (1985, 1990s)

Assume class of Woodin cardinals. Suppose $A \subseteq \mathbb{R}$ is a universally Baire set, g is V-generic and h is $V[g]$-generic. Then there is an elementary embedding

$$j : L(A_g, \mathbb{R}_g) \to L(A_{g*h}, \mathbb{R}_{g*h}).$$

such that $j(A_g) = A_{g*h}$.

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- We now want to see if they satisfy CH.

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Definition: The Perfect Set Property, PSP

A set is said to have the **perfect set property** if it is either countable or contains a non-empty perfect subset.

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Theorem: Bernstein (1908)

The Axiom of Choice implies that there is a set of reals without the perfect set property.

Games people play

Let $A \subseteq \omega^\omega$ be a fixed set of real numbers (the "payoff set").

Two players, **Player I** and **Player II**, play an infinite game G_A of perfect information. They take turns choosing natural numbers $a_n \in \omega$.

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Definition:

We say $A \subseteq \mathbb{R} = \omega^\omega$ is **determined** if one of the players has a winning strategy in G_A.

History of PSP and Determinacy



1884, Cantor: Closed sets have the Perfect Set Property (PSP).

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AD **and** CH

Thus, AD implies Version 1 of CH.

Beyond AD

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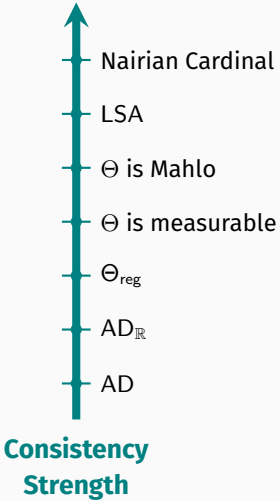
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Θ is the least ordinal that is not a surjective image of the reals.

The Hierarchy of Determinacy Axioms

- AD.
- $\mathrm{AD}_{\mathbb{R}}$.
- $\Theta_{\mathrm{reg}} = \mathrm{AD}_{\mathbb{R}} + \Theta$ is regular.
- $\mathrm{AD}_{\mathbb{R}} + \Theta$ is a measurable cardinal.
- $\mathrm{AD}_{\mathbb{R}} + \Theta$ is a Mahlo cardinal.
- LSA (The Largest Suslin Axiom).
- Θ is a Nairian Cardinal.
- ...



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Assume there is a Woodin cardinal that is a limit of Woodin cardinals. Then there is a transitive model M containing the reals and ordinals such that $M \vDash \Theta_{\mathsf{reg}}$.

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- The set-theoretic universe under AD and its extensions is rich in mathematical structures.
- AD and its extensions imply the existence of all sorts of ultrafilters, Ramsey theoretic properties, a complete analysis of regularity properties, and much more.
- The study of models of AD is still going on, and many questions remain open.

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- The resulting hierarchies are so deeply intertwined that it is only a matter of taste (or insistence on AC) that can be used to distinguish the two.
- In this talk, we will focus on the AD side.

Towards Version 2 of CH.

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- Since PD is true, there cannot be a projective $\aleph_1$-sequence of distinct reals.
- We need to find axiomatic systems that come with a lot of structure.

Forcing Axioms

> **Menachem Magidor**
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- Forcing axioms generalize this topological principle to much larger collections of dense sets (e.g., $\aleph_1$ many) and broader classes of partial orders.

Martin's Maximum (MM)

Definition: Martin's Maximum (Foreman, Magidor, and Shelah, 1988)

MM states that for any partial order $\mathbb{P}$ that preserves stationary subsets of ω_1, and for any collection $\mathcal{D}$ of $\aleph_1$ dense open subsets of $\mathbb{P}$, there exists a $\mathcal{D}$-generic filter, i.e., a filter $g \subseteq \mathbb{P}$ that intersects all members of $\mathcal{D}$.

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Theorem: Consistency of MM (Foreman, Magidor, and Shelah, 1988)

Suppose there is a supercompact cardinal. Then ZFC + MM holds in a forcing extension of V (that preserves the stationary subsets of ω_1).

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- (Moore, 2006) There is a five-element basis for the class of uncountable linear orders.

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- The study of Forcing Axioms is one of the most active areas of set theory, and many questions remain open.

Putting it all together

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- However, only Determinacy Axioms and Forcing Axioms have been studied extensively as theories.

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- Unfortunately, Determinacy Axioms imply CH (Version 1) and Forcing Axioms imply $\neg$CH.
- Which one is the right one?
- Nothing short of an inconsistency proof will ever determine the winner in objective terms.

Steel's Program

> **Thesis**
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Thesis

All sufficiently rich foundational frameworks are bi-interpretable.

Given any two natural foundational frameworks T and S, we can construct natural models of T from models of S (and vice versa) via canonical methods.

Definition: MM(c)

$\mathrm{MM}(c)$ is the statement that Martin's Maximum holds for the stationary set preserving posets of the size of the continuum.

Bi-interpreting: from Determinacy to Forcing Axioms

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Theorem: Woodin, (1990s)

Assume $V = L(\mathscr{P}(\mathbb{R})) + \mathsf{AD}_{\mathbb{R}} +$ "Θ is a regular cardinal". Let G be generic for $\mathbb{P}_{\mathsf{max}} * \mathsf{Add}(\omega_3, 1)$. Then $V[G] \vDash \mathsf{MM(c)}$.

Theorem: Steel, (2005)

Assume MM. Then AD holds in $L(\mathbb{R})$.

Bi-interpreting: from Forcing Axioms to Determinacy

Theorem: Steel, (2005)

Assume MM. Then AD holds in $L(\mathbb{R})$.

Theorem: S.-Trang, (2015)

Assume MM. Then there is a transitive definable M containing all the reals and ordinals such that $M \vDash$ LSA.

Major obstacle

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- Up until recently, all known models of determinacy were of the form $L(\mathscr{P}(\mathbb{R}))$.
- Provably, we cannot force MM over such models.
- The issue is that models of the form $L(X)$ where X is a set are highly non-compact beyond X (as mentioned before).
- We need determinacy models with rich structure at every level.

Nairian Models

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- Nairian Models are not of the form "L of a set".
- No reason why they should be of any interest.

Nairian Models are interesting

Theorem: Steel and Woodin, 2023-2024

Assume $V = L(\mathscr{P}(\mathbb{R})) +$ LSA $+$ HPC. Suppose Θ is a limit of Woodin cardinals in HOD and let N be the Nairian Model at Θ. Then $N \vDash$ ZF.

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Assume $V = L(\mathscr{P}(\mathbb{R})) + \mathsf{LSA} + \mathsf{HPC}$. Suppose Θ is a limit of Woodin cardinals in HOD and let N be the Nairian Model at Θ. Then $N \vDash \mathsf{ZF}$.

Theorem: Blue, Larson and S., 2024-2025

Assume the existence of a Woodin cardinal that is a limit of Woodin cardinals. Then the theory

$$\mathsf{ZFC} + \mathsf{MM}(\mathfrak{c}) + \forall \kappa \geq \omega_1 (\square_\kappa \text{ fails})$$

can be forced over a Nairian Model of some determinacy model.

Conjecture

MM can be forced over Nairian Models defined inside models of determinacy.

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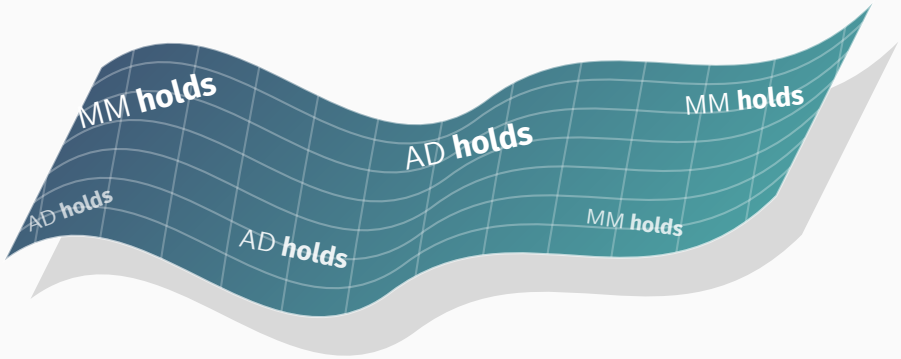
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- Forcing Axioms to Determinacy Axioms will be the topic of the third talk.

A View of the Set-Theoretic Universe



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- The structures we choose to study lead us to our favorite foundational framework that gives the answers we want.

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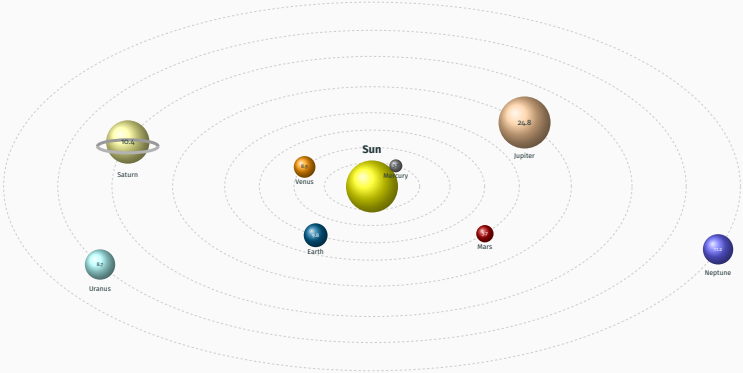
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- But we can also think in a unified way. Some questions have better answers in other frameworks.
- So we should work on unifying our frameworks.

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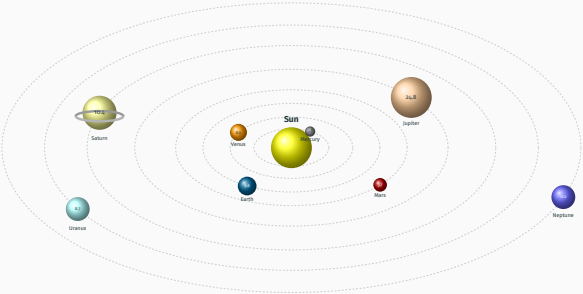
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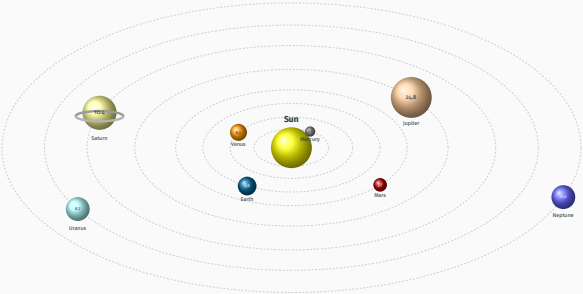
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- What is important is to have a way of traveling around the universe.

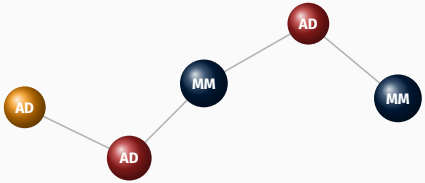


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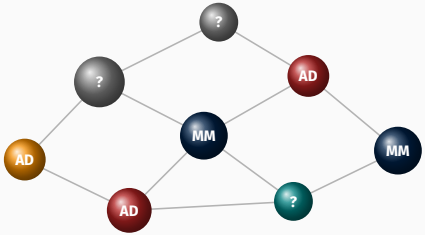
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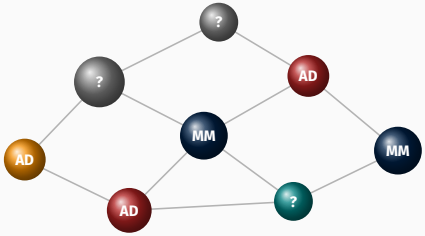
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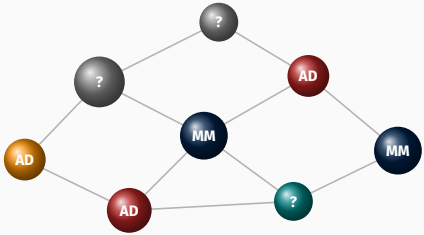
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- But (unfortunately) CH is not one of those.



PD holds everywhere


Thank You
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