Math W54 Midterm 2 July 26, 2022

Name:

Total: **50 points.**

Duration: **2 hours.**

- This is Midterm 2. The exam is out of 50 points and has 5 questions, where each question is worth 10 points. You are required to solve **all** questions. Some of these questions have multiple parts; next to each part, you will find the amount of points the question is worth.
- You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by explanation or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- **You must take this exam completely alone.** Showing it to or discussing it with anyone else is forbidden. Reproducing or sharing this material on any other websites or platforms is also forbidden. You are permitted to consult your notes, the textbook, any materials handed out in class, and any materials on the bCourses site. You may not consult any other resources (for instance, public websites, Wolfram Alpha, and calculators).

1. (a) (5 points) Consider the vector $\mathbf{w} = (1/\sqrt{2}, 1/\sqrt{2})^T$ in $\mathbb{R}^2$. Find a unit vector $\mathbf{v}$ such that the angle from $\mathbf{w}$ to $\mathbf{v}$ is $\pi/3$.

Proof. Let $\mathbf{v} = (x, y)^T$. Since the angle from $\mathbf{w}$ to $\mathbf{v}$ is $\pi/3$, and since $\mathbf{w}$ and $\mathbf{v}$ are unit vectors, we have

$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||||\mathbf{w}|| \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

So x, y must satisfy $y = \sqrt{2}/2 - x$. To ensure $\mathbf{v}$ is a unit vector, we must have $x^2 + y^2 = 1$. So

$$0 = x^2 - \frac{\sqrt{2}}{2}x - \frac{1}{4}$$

holds. It follows,

$$x = \frac{\sqrt{2} \pm \sqrt{6}}{4}.$$

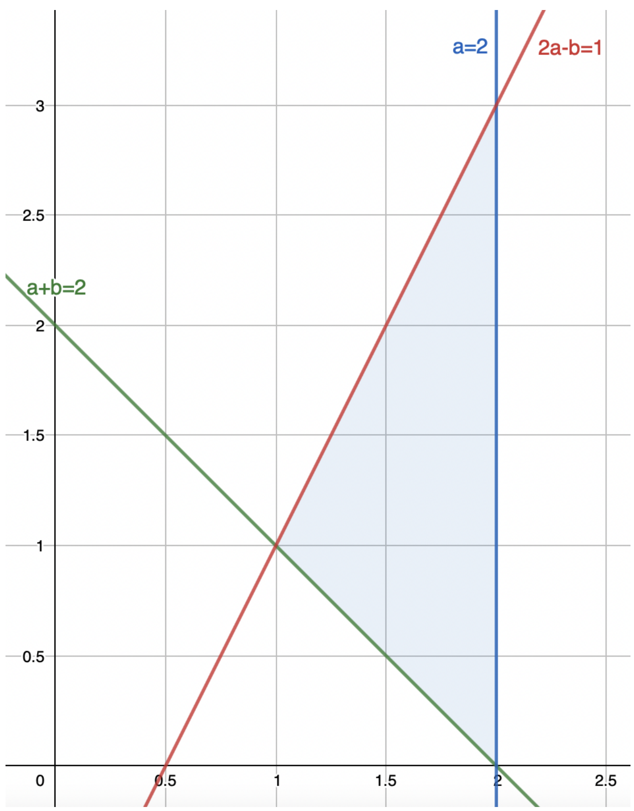
We want the angle from $\mathbf{w}$ to $\mathbf{v}$ to be $\pi/3$, so we must have $x = (\sqrt{2} - \sqrt{6})/4$ and $y = (\sqrt{2} + \sqrt{6})/4$. □

(b) (5 points) Find the set of all vectors v in $\mathbb{R}^2$ such that $v \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq 2$, $v \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \leq 2$, and $v \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} \geq 1$. Draw a picture in $\mathbb{R}^2$ to describe the set you find.

Solution. Suppose $v = \begin{bmatrix} a \\ b \end{bmatrix}$. We need:

$$v \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a + b \geq 2$$
$$v \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \leq 2$$
$$v \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2a - b \geq 1$$

In $\mathbb{R}^2$, the set of vectors satisfying these three conditions are exactly the ones in the area bounded by the three lines $a + b = 2$, $a = 2$, and $2a - b = 1$, i.e. the triangle below.



□

2. Show that if A is the matrix of an orthogonal projection of $\mathbb{R}^n$ onto a subspace W, then A is diagonalizable.

   Solution. A is diagonalizable if and only if we can find a basis of $\mathbb{R}^n$ consisting of eigenvectors of A, which is what we will do. Pick an orthogonal basis $w_1, \ldots, w_k$ of W, and complete it to an orthogonal basis of $\mathbb{R}^n$ by adding $v_{k+1}, \ldots, v_n$. Since $w_i \in W$, $Aw_i = w_i$, so w_i is an eigenvector with eigenvalue 1. Since $v_i \in W^\perp$, $Av_i = 0$, so v_i is an eigenvector of A with eigenvalue 0. $\square$

3. Consider the vector space $\mathbb{P}_2$ of polynomials of degree at most 2. Let $S : \mathbb{P}_2 \to \mathbb{R}^3$ be the isomorphism defined by

$$1 \mapsto \mathbf{e}_1,\ t \mapsto \mathbf{e}_2,\ t^2 \mapsto \mathbf{e}_3.$$

(This is the standard coordinate representation of $\mathbb{P}_2$.)

Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by

$$T(p(t)) = p(1-t),$$

i.e., T changes variable from t to $1-t$.

(a) (3 points) Compute the standard matrix A for the transformation STS^{-1}.

Proof. By definition, we have $T(1) = 1$, $T(x) = 1 - x$, and $T(x^2) = 1 - 2x + x^2$. Conjugating by S, it follows that $STS^{-1}(\mathbf{e}_1) = \mathbf{e}_1$, $STS^{-1}(\mathbf{e}_2) = \mathbf{e}_1 - \mathbf{e}_2$, and $STS^{-1}(\mathbf{e}_3) = \mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$. So the standard matrix for STS^{-1} is given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\square$

(b) (3 points) Calculate the eigenvalues of A.

Proof. A is upper triangular, so its eigenvalues are 1 and -1 with algebraic multiplicities 2 and 1, respectively. $\square$

(c) (4 points) Does $\mathbb{R}^3$ admit a basis which represents STS^{-1} as a diagonal matrix? If so, exhibit one. If not, explain why.

Proof. The condition in question holds if, and only if, $\mathbb{R}^3$ admits a basis of eigenvectors for STS^{-1}, which in turn holds if, and only if, the geometric and algebraic multiplicities of each of its eigenvalues coincide.

Thus, we begin by calculating bases for the eigenspaces E_1 and E_{-1}, i.e. the kernels of $A - I$ and $A + I$, respectively. After row-reducing these matrices, we find that the general solution of $(A - I)(\mathbf{x}) = 0$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

and the general solution of $(A + I)(\mathbf{x}) = 0$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}.$$

So $(1, 0, 0)^T, (0, -1, 1)^T$ is a basis for E_1 and $(-1, 2, 0)^T$ is a basis for E_{-1}. In particular, the geometric multiplicities of the eigenvalues 1 and -1 are 2 and 1, respectively. These coincide with their respective algebraic multiplicities, so we conclude that $\mathbb{R}^3$ admits a basis which represents STS^{-1} as a diagonal matrix—namely, the (ordered!) basis $(1, 0, 0)^T, (-1, 2, 0)^T, (0, -1, 1)^T$. □

4. (10 points) Given similar matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 \\ 7 & 9 & 8 \\ 4 & 6 & 5 \end{bmatrix}$. Find a basis in $\mathbb{R}^3$ where the matrix for A is B.

Solution 1. Observe that we get B by swapping the last two rows and the last two columns of A. In other words, B with its last two rows swapped = A with its last two columns swapped. To swap the last two rows of B, we can multiply the elementary invertible matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

on the left of B. Now if we multiply this P on the right of A, it will swap the last two columns of A. So we have:

$$PB = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix} = AP$$

So $B = P^{-1}AP$, and the columns of P is the basis we want. □

Solution 2. We know $B = P^{-1}AP$ for some invertible matrix P and we want to find the columns of P. Let

$$P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

We have $PB = AP$, or:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 7 & 9 & 8 \\ 4 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

This gives us a system of 9 equations in 9 variables, so we can solve for $a, b, c, d, e, f, g, h, i$. The basis will be the three column vectors of P. (**On the exam you need to explicitly find the basis**) □

5. An $n \times n$ matrix N is said to be *nilpotent* if $N^k = 0$ for some $k \in \mathbb{N}$.

(a) (6 points) Prove that $I - N$ is invertible by finding $(I - N)^{-1}$ (*Hint:* Think of an analogue to the series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ from calculus).

Solution 1. Based on the hint, we consider the series

$$Q = I + N + N^2 + \cdots + N^k + N^{k+1} + \cdots$$

of matrices as a candidate for $(I - N)^{-1}$. Fortunately, since $N^k = 0$, we have $N^\ell = 0$ for all $\ell \geq k$, so the series is not really infinite (no issues of convergence). Thus we have $Q = I + N + N^2 + \cdots + N^{k-1}$ as a candidate for the inverse. Indeed, multiplying the matrices,

$$\begin{aligned}
(I - N)Q &= (I - N)(I + N + N^2 + \cdots + N^{k-1}) \\
&= (I + N + N^2 + \cdots N^{k-1}) - (N + N^2 + \cdots + N^k) \\
&= I + (N - N) + (N^2 - N^2) + \cdots (N^{k-1} - N^{k-1}) - N^k \\
&= I
\end{aligned}$$

since $N^k = 0$. Thus $I - N$ is invertible with inverse Q. □

Solution 2. Note that $I - N^k = I$ since $N^k = 0$. Then factorizing the left hand side, $(I - N)(I + N + N^2 + \cdots N^{k-1}) = I$, yielding the result.

□

(b) (4 points) Suppose A is an $n \times n$ nilpotent matrix such that $A^2 = 0$. Prove that $\operatorname{rank}(A) \leq \frac{n}{2}$.
(*Hint:* What relationship can you derive between $\operatorname{col}(A)$ and $\operatorname{nul}(A)$ based on the fact that $A^2 = 0$?).

Solution. Let $\mathbf{b} \in \operatorname{col}(A)$. Then we know that there is $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$. Now left-multiplying by A, we get that $A^2\mathbf{x} = A\mathbf{b}$. However, since $A^2 = 0$, we know that $A^2\mathbf{x} = \mathbf{0}$, so we get that $A\mathbf{b} = \mathbf{0}$. Then $\mathbf{b} \in \operatorname{nul}(A)$ necessarily. Subsequently, we can conclude that $\operatorname{col}(A) \subseteq \operatorname{nul}(A)$, or $\operatorname{rank}(A) \leq \operatorname{null}(A)$. Now by the rank-nullity theorem,

$$\operatorname{rank}(A) + \operatorname{null}(A) = n$$

However, since $\operatorname{rank}(A) \leq \operatorname{null}(A)$, we can conclude that

$$2\operatorname{rank}(A) \leq \operatorname{rank}(A) + \operatorname{null}(A) = n, \text{ so } \operatorname{rank}(A) \leq \frac{n}{2}$$

as desired. $\square$