ZO OM

Math 110
Professor Kenneth A. Ribet
Second Midterm April 1, 2020

Problem	0	1	2	3	4	5	6	7	Total
Points	1	5	6	7	7	7	6	1	40

This is an open book unproctored online exam. Please write carefully and clearly, *using sentences* (not just symbols). Remember that the paper you hand in will be your only representative when your work is graded.

Recall that **F** denotes the real or complex field in our course.

You have 90 minutes (hard limit) to work on the exam and 15 minutes to upload your work to `Gradescope`. You may consult the textbook, all the material on `bCourses`, the class `piazza` and your own notes. Not permitted: online searches, other uses of the internet, collaboration with other people (electronic or otherwise).

Your name:

Your SID:

Your GSI's name:

Time in Berkeley when you started on the exam:

Time in Berkeley when you finished working on the exam:

0. You were able to get 1 point for successfully uploading your "solutions" to the practice exam by March 31.

1. Use the Cauchy–Schwarz inequality to establish the inequality

$$x + 4y + 8z \leq 9$$

if x, y and z are real numbers satisfying

$$x^2 + y^2 + z^2 = 1.$$

2. Consider $\mathcal{P}_2(\mathbf{R})$ with the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx.$$

Find a basis for the orthogonal complement of $\mathcal{P}_1(\mathbf{R})$ in this space.

3. Suppose that $T \in \mathcal{L}(V)$ is an operator on a finite-dimensional real vector space and that $T^3 = T$.

a. Explain why 0, -1 and 1 are the only possible eigenvalues of T.

b. Show that at least one of 0, -1 and 1 is an eigenvalue if the vector space is nonzero.

4. Let U and X be subspaces of a vector space V over $\mathbf{F}$. Let $\pi : V \to V/U$ be the quotient map and let $S : X \to V/U$ be the restriction of π to X. (Thus $Sx = \pi x$ for $x \in X$.) Establish that:

(a) S is injective if and only if $X \cap U = \{0\}$;

(b) S is surjective if and only if $V = X + U$;

(c) S is an isomorphism of vector spaces if and only if V is the direct sum of U and X.

5. Let $S \in \mathcal{L}(V)$ be an operator on a finite-dimensional $\mathbf{F}$-vector space. Let $v \in V$ be a nonzero vector and let k be the largest positive integer for which the list

$$v = S^0 v, Sv, S^2 v, \ldots, S^{k-1} v$$

is linearly independent.

a. Prove that

$$W = \operatorname{span}(v, Sv, \ldots, S^{k-1} v)$$

is a S-invariant subspace of V.

b. If U is a S-invariant subspace of V that contains v, show that U contains W.

6. Let $T \in \mathcal{L}(V)$ be a linear operator on V and let U be a T-invariant subspace of V. Suppose that $v_1, \ldots, v_k$ are elements of V such that

$$Tv_j = \lambda_j v_j, \quad j = 1, \ldots, k,$$

where the λ_j are distinct elements of $\mathbf{F}$. If

$$(v_1 + U) + (v_2 + U) + \cdots + (v_k + U) = 0 + U \in V/U,$$

show that $v_j + U = U$ for all $j = 1, \ldots, k$.

7. At the conclusion of the exam, please copy and sign the following statement:

> "*As a member of the UC Berkeley community, I acted with honesty, integrity, and respect for others during this exam. The work that I am uploading is my own work. I did not collaborate with or contact anyone during the exam, search online for problem solutions, or otherwise violate the instructions for this examination.*"