LET THERE BE BLOCKS

Math 110

Professor Kenneth A. Ribet

Final Exam May 15, 2020

3–6PM

Problem	1	2	3	4	5	6	7	8	9	10	Total
Points	8	8	8	8	8	8	8	8	8	2	66

This is an open book unproctored exam. Please write carefully and clearly, *in complete English sentences*. The paper you upload will be your only representative when your work is graded.

Do only 8 of the 9 mathematical problems. Decide which problem you will not do and make your choice clear by writing on each problem "Doing this problem" or "Not doing this problem." Please write solutions on different sheets of paper; in other words, two problem solutions shouldn't be on the same page.

Write explicitly at the top and bottom of your exam a statement of the form "I didn't do problem X."

Please write your name, your SID and your GSI's name on each page that you submit.

You have 180 minutes (hard limit) to work on the exam and 15 minutes to upload your work to `Gradescope`. You may consult the textbook, all the material on `bCourses`, the class `piazza` and your own notes.

Not permitted: online searches, other uses of the internet, collaboration with other people (electronic or otherwise). Please act with honesty, integrity and respect for others.

Recall that $\mathbf{F}$ denotes the real or complex field. All vector spaces are finite-dimensional unless otherwise specified.

1. Let $T : V \to W$ be a linear map. Show that

$$S : V/(\operatorname{null} T) \longrightarrow \operatorname{range} T, \qquad S(v + \operatorname{null} T) = Tv$$

is a well-defined linear map from $V/(\operatorname{null} T)$ to $\operatorname{range} T$. Prove that S is an isomorphism and deduce that

$$\dim(V/(\operatorname{null} T)) = \dim(\operatorname{range} T).$$

What is the relation between this formula and the rank–nullity formula

$$\dim V = \dim(\operatorname{null} T) + \dim(\operatorname{range} T)?$$

2. Let $S : V \to W$ be a linear map, and let $S' : W' \to V'$ be the dual map of S. Imagine that your friend Oski asks you to outline an argument that S and S' have the same rank. (The coincidence of ranks is often described as the equality between row and column ranks of a matrix.) What outline would you provide to your friend? Be sure that your summary omits none of the major steps in the argument that you are summarizing—Oski may be cute and friendly, but he's still a bear.

3. Let W be a subspace of the $\mathbf{F}$-vector space V with $W \neq V$. Show that there is a nonzero linear map $T : V \to \mathbf{F}$ whose restriction to W is 0.

4. Let a and b be real numbers, and let A be the $n \times n$ matrix whose diagonal entries are a and non-diagonal entries are b. Find the eigenvalues of the linear operator on $\mathbf{R}^n$ defined by $x \mapsto Ax$. (Be sure to explain your calculations and reasoning in full English sentences.)

5. Suppose that $N \in \mathcal{L}(V)$ is nilpotent and that there are vectors $v_1, \ldots, v_4$ such that

$$N^5 v_1, N^4 v_1, \ldots, N v_1, v_1, \;\; N v_2, v_2, \;\; v_3, \;\; N^3 v_4, N^2 v_4, N v_4, v_4$$

is a basis of V, and such that

$$N^6 v_1 = N^2 v_2 = N v_3 = N^4 v_4 = 0.$$

Fill in the table:

i	1	2	3	4	5	6	7	8	$\cdots$
$\dim \operatorname{null} N^i$									$\cdots$.

Be sure to explain your reasoning in complete English sentences.

6. Let T be an operator on a complex vector space V such that

$$\text{null}(T - \alpha I)^2 = \text{null}(T - \alpha I)$$

for all complex numbers α. Show that there is a basis of V consisting of eigenvectors of T.

7. Let U be a subspace of the $\mathbf{F}$-vector space V, with U neither $\{0\}$ nor V. Let $v_1, \ldots, v_n$ be a basis of V and consider the list

$$(*) \qquad v_1 + U, \ldots, v_n + U$$

of vectors in V/U.

a. Explain why it is possible to obtain a basis of V/U by removing some vectors from the list $(*)$.

b. After possibly re-ordering the original basis, we can and will suppose that

$$v_1 + U, \ldots, v_d + U$$

is a basis of V/U. Show that V is the direct sum of U and $\text{span}(v_1, \ldots, v_d)$.

8. Suppose T is a positive operator on an inner product space. Prove that T and $\sqrt{T}$ have the same null space and also that they have the same rank. Is it true that T and $\sqrt{T}$ have the same range? (Give a proof that they do or describe a situation where they do not.)

9. Suppose that V is an inner product space over $\mathbf{R}$ and that $T \in \mathcal{L}(V)$ is a normal operator that has an upper-triangular matrix with respect to some basis of V. Prove that T is self-adjoint.

Please write explicitly at the top and bottom of your exam a statement of the form "I didn't do problem X."

10. At the conclusion of the exam, please copy and sign the following statement:

> "*As a member of the UC Berkeley community, I acted with honesty, integrity, and respect for others during this exam. The work that I am uploading is my own work. I did not collaborate with or contact anyone during the exam. I did not obtain solutions from* `chegg.com` *or other sites. I adhered to all instructions for this examination.*"