PROFESSOR KENNETH A. RIBET

Last Midterm Examination

March 18, 2010

2:10–3:30 PM, 10 Evans Hall

Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. For numerical questions, *show your work* but do not worry about simplifying answers. For proofs, write your arguments in complete sentences that explain what you are doing. Remember that your paper becomes your only representative after the exam is over.

Problem	Possible points
1	6 points
2	12 points
3	6 points
4	6 points
Total:	30 points

1. a. Use row operations to find the inverse of the matrix $\begin{pmatrix} -2 & 1 & 0 \\ 4 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

I'm sure that all of you know how to do this and that most of you will do it correctly. The answer seems to be $\begin{pmatrix} 2 & 1 & 1 \\ 5 & 2 & 2 \\ 7 & 3 & 2 \end{pmatrix}$.

b. Let A be an $m \times n$ matrix of rank m and let B be an $n \times p$ matrix with rank n. Determine the rank of AB. Prove that your answer is correct.

Think of $L_A : F^n \to F^m$ and $L_B : F^p \to F^n$. Their ranks are equal to the dimensions of the spaces to which they are mapping. Thus these maps are *onto*. It follows that the same statement is true for $L_A \circ L_B = L_{AB}$. In other words, AB has rank m.

2. Label each of the following statements as TRUE or FALSE. Along with your answer, provide a counterexample, an informal proof or an explanation.

a. The determinant is a linear function $\mathrm{M}_{n\times n}(F) \to F$.

The best answer is "FALSE." Indeed, the determinant is a linear function of each of its rows (or colums), but it is not a linear function in general. For example, if you multiply A by a scalar c, $\det(A)$ gets multiplied by c^n.

b. Every $n \times n$ matrix may be written as a product of elementary matrices.

This is true for invertible matrices but is FALSE in general.

c. Let V be a vector space over the field of complex numbers. If $\{ f_1, f_2, f_3 \}$ is the basis of V^* dual to $\{ v_1, v_2, v_3 \}$, then $\{ f_1, 2f_2, 3f_3 \}$ is the basis dual to $\{ v_1, 2v_2, 3v_3 \}$.

As discussed in the review lecture on March 16, when you change a basis of V by a matrix Q, the dual basis changes by the transpose of the inverse of Q. Here, Q is the diagonal matrix with diagonal entries 1, 2 and 3; the dual basis changes by the diagonal matrix with entries 1, 1/2 and 1/3. Accordingly, the basis dual to $\{ v_1, 2v_2, 3v_3 \}$ is $\{ f_1, (1/2)f_2, (1/3)f_3 \}$, so the statement is FALSE.

d. If $Ax = 0$ has exactly one solution, then $Ax = b$ has exactly one solution.

How to interpret this? There is no statement that A is square, so we have to think that A might be rectangular in some way. The hypothesis states in other terms that L_A is 1-1, but L_A might not be onto if A has more rows than columns. Hence the statement is FALSE. To create a specific counterexample, imagine that A is 2×1; then it represents two equations in one unknown! These equations might be inconsistent if the two entries in b are not equal to each other—in this case, there will be no solution.

e. One may find positive integers n and m and matrices $A \in \mathrm{M}_{n\times m}(C)$, $B \in \mathrm{M}_{m\times n}(C)$ such that $AB = I_n$ and $BA = 0_m$.

This problem was suggested by the GSIs, who proposed the following quick solution, which is based on the well known fact that AB and BA have equal traces: If A and B are as in the statement of the problem, then AB has trace n and BA has trace 0. Since $n \neq 0$ in C, there is no situation like this, so the statement is FALSE.

Another way to see this: If A and B are as in the statement of the problem, consider ABA. This is $A \cdot 0_m = 0_{n\times m}$ and also $I_n \cdot A = A$. Hence $A = 0_{n\times m}$, so that AB can't be the identity.

f. If $T : V \to W$ is a linear transformation between finite-dimensional vector spaces and $v_1, v_2, \ldots, v_k$ are linearly independent vectors of V, then $T(v_1), T(v_2), \ldots, T(v_k)$ may be extended to a basis of W.

This is clearly FALSE because $T(v_1), T(v_2), \ldots, T(v_k)$ might happen to be linearly dependent and those won't be part of a basis.

Amazing, everything is FALSE today!

3. Let V be the real vector space $\mathcal{P}_3(\mathbf{R})$ of polynomials of degree ≤ 3 with real coefficients. Let $\beta = \{1, x, x^2, x^3\}$ be the standard ordered basis of V. Let $T : V \to V^*$ be the linear transformation that sends $p \in V$ to f_p, where $f_p(q) = \displaystyle\int_{-1}^{1} p(x)q(x)\,dx$. Find $[T]_\beta^{\beta^*}$, where β^* is the dual basis of β.

See problem 4 on the second midterm from my course in spring, 2005:
http://math.berkeley.edu/~ribet/110/mt2sols.pdf.
Here the integral is from -1 to 1 instead of 0 to 1; that should make the matrix have lots of zeros this time.

4. Let V be a finite-dimensional vector space over a field and let W be a subspace of V. Suppose that v is a vector of V that does not lie in W. Show that there is an element f of V^* such that $f(v) = 1$ and $f(w) = 0$ for all $w \in W$.

Take a basis $\{ w_1, \ldots, w_k \}$ of W and note that the set $\{ w_1, \ldots, w_k, v \}$ is still linearly independent because v is not in the span of the w_i. Complete this list to get an ordered basis of all of V: $\{ v_1, \ldots, v_n \}$, where $v_1, \ldots, v_k$ are simply $w_1, \ldots, w_k$ and $v_{k+1} = v$. Let $\{ f_1, \ldots, f_n \}$ be the dual basis of V^* and let $f = f_{k+1}$. Then $f(v) = 1$ because $v = v_{k+1}$ and $f = f_{k+1}$. Also, $f(v_j) = 0$ if $j \neq k + 1$. In particular, $f(w_i) = 0$ for $i = 1, \ldots, k$. Hence f is 0 on the span of the w_i, which is W.