Aspects of the Turing Jump

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1 Introduction

Definition 1.1 The *Turing Jump* is the function which maps a set $X \subseteq \mathbb{N}$ to X', the halting problem relative to X. Fixing a recursive enumeration of all Turing machines,

$$X' = \{e : \text{The } e\text{th Turing machine with oracle } X \text{ halts.}\}$$

X' is the canonical example of a set which is definable from X but not recursive in X. The Turing degree of X' depends only on the Turing degree of X, so the jump induces an increasing function on the Turing degrees $\mathcal{D}$.

In this paper, we will discuss two aspects of the jump and its iterations. First, we will show that they are implicitly characterized by general properties of relative definability. Second, we will present the Shore and Slaman [1999] theorem that the function $x \mapsto x'$ is first order definable in the Turing degrees. Finally, we will pose analogous questions about the relation *y is recursively enumerable in x* and discuss what is known about them.

Our discussion will rest on two technical facts, which are generalizations of the following two theorems.

Theorem 1.2 (Friedberg [1957]) *Suppose that $x \geq_T 0'$. Then there is a g such that $g' = x$.*

In other words, every sufficiently complicated degree is the jump of some degree.

Theorem 1.3 (Posner and Robinson [1981]) *Suppose that $0 \not\geq_T x$. Then there is a g such that $x \vee g = g'$.*

Similarly, every nontrivial degree is the jump relative to some other degree.

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1.1 Hierarchies of definability

The arithmetic and hyperarithmetic hierarchies provide the context in which we can extend the inversion and join theorems.

The arithmetic hierarchy.

Definition 1.4 1. Σ^0_0 and Π^0_0 denote the set of bounded formulas in first order arithmetic.

2. Σ^0_{n+1} denotes the set of formulas $(\exists n_1, \ldots, n_k)\psi$, where $\psi \in \Pi^0_n$.

3. Π^0_{n+1} denotes the set of formulas $(\forall n_1, \ldots, n_k)\psi$, where $\psi \in \Sigma^0_n$.

Definition 1.5 1. $X \subseteq \mathbb{N}$ is Σ^0_n or Π^0_n if and only if it is definable in arithmetic by a formula of the corresponding type.

2. $X \subseteq \mathbb{N}$ is Δ^0_n if and only if it is both Σ^0_n and Π^0_n.

As is known very well, X is Δ^0_{n+1} if and only if $X \leq_T \emptyset^{(n)}$. Thus, the arithmetic sets are generated from the empty set by applying the jump and recursive functionals.

The hyperarithmetic hierarchy. Davis [1950] extended the arithmetic hierarchy into the transfinite by iterating the jump along recursive well-orderings of $\mathbb{N}$. At limits λ, he used the recursive presentation of λ to form the recursive join of the sets associated (by that presentation) to ordinals less than λ.

For example, $\emptyset^{(\omega)} = \{(n,m) : m \in \emptyset^{(n)}\}$ represents the ωth jump of the empty set.

Spector [1955] showed that any two sets associated with the same recursive ordinal have the same Turing degree. Consequently, we have degree invariant functions $X \mapsto X^{(\alpha)}$. Of course, we can define the hyperarithmetic hierarchy relative to a presentation of any countable ordinal, and then we obtain a degree invariant function $X \mapsto X^{(\alpha)}$ which is defined on those X's which compute that presentation of α.

1.1.1 Jockusch–Shore *REA*-operators

Definition 1.6 (Jockusch and Shore [1984]) An *REA-operator* is a function J from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$ such that there is an e such that for all X, $J(X)$ is the join of X with the eth set which is recursively enumerable relative to X (in a fixed recursive enumeration of all the relativized recursive enumerations).

An α-*REA*-operator is an α-length iteration of *REA*-operators, where the iteration is organized using a recursive presentation of α as in the hyperarithmetic hierarchy. For example, the canonical 2-*REA* operator is the map $A \mapsto A''$. But, note that not every 2-*REA* operator is degree invariant.

1.2 Extensions of the prototypes

Surprisingly, the inversion and join properties of the Turing jump are shared by all α-*REA* operators.

Theorem 1.7 (Jockusch and Shore [1984]) *Suppose that $C \geq_T \emptyset^{(\alpha)}$ and J is an α-REA operator. There is a G such that $J(G) \equiv_T C$.*

Thus, every sufficiently complicated degree is in the range of J.

Theorem 2.1 (Shore and Slaman [1999]) *Suppose that J is an α-REA operator and for all $\beta < \alpha$, $\emptyset^{(\beta)} \not\geq_T X$. There is a G such that $X \oplus G \equiv_T J(G)$.*

Similarly, every nontrivial set represents the value of J relative to some set. We will outline the proof of Theorem 2.1 in Section 2. We should note that there is a history behind it, with substantial preliminary results by Jockusch and Shore and by Cooper. A full account is given in [Shore and Slaman, 1999].

We can interpret Theorems 1.2 and 1.3, and their generalizations to Theorems 1.7 and 2.1, as asserting a fundamental role for the jump and its iterations. By Theorem 1.3, the only way by which a set can be not recursive is by being equivalent to the jump relative to some other set. And by Theorem 2.1, the same phenomenon is repeated through the transfinite. The applications which follow can be viewed as realizations of this interpretation.

1.3 Abstract closure operators

In our first application, we argue that the iterations of the Turing jump have a special role within relative definability. In the following definition, we are thinking of M as a function which maps X in $2^{\mathbb{N}}$ to the set of those Y's in $2^{\mathbb{N}}$ which are definable from X (in some specific way).

Definition 1.8 A *closure operator* is a map $M : 2^{\mathbb{N}} \to 2^{2^{\mathbb{N}}}$ with the following properties.

1. For all $X \in 2^{\mathbb{N}}$, $X \in M(X)$.
2. For all X and for all Z, if Z is recursive in finitely many elements of $M(X)$, then $Z \in M(X)$. *$M(X)$ is closed under join and relative computation.*
3. For all X and Y in $2^{\mathbb{N}}$, if X is recursive in Y then $M(X) \subseteq M(Y)$. *M is monotone.*

The functions mapping X to the collection of subsets of $\mathbb{N}$ recursive in X, recursive in X', or arithmetically definable from X are closure operators.

Comparing closure operators.

Definition 1.9 If M_1 and M_2 are closure operators, then $M_1 \leq M_2$ if and only if there is a set of natural numbers B, such that for all X, if B is recursive in X, then $M_1(X) \subseteq M_2(X)$.

In other words, $M_1 \leq M_2$ if and only if for all sufficiently complicated X, $M_1(X) \subseteq M_2(X)$. We say M_1 and M_2 are *equivalent* when for all sufficiently complicated X, $M_1(X) = M_2(X)$.

Remark 1.10 This notion is a $2^{\mathbb{N}} \to 2^{2^{\mathbb{N}}}$ variation on Martin's ordering of degree invariant functions on reals. See [Kechris and Moschovakis, 1978] for further information on the Martin order and on Martin's conjectures concerning that order.

1.3.1 Implicitly characterizing the hyperarithmetic hierarchy

With the following theorem, we give a complete classification of the Borel closure operators, up to equivalence.

Theorem 3.1 *If M is a closure operator such that the relation $Y \in M(X)$ is Borel, then one of the following conditions holds.*

1. *There is a countable ordinal α such that M is equivalent to the map*
$$X \mapsto \{Y : Y \text{ is recursive in } X^{(\alpha)}\}.$$
2. *There is a countable ordinal α such that M is equivalent to the map*
$$X \mapsto \{Y : (\exists \beta < \alpha)[Y \text{ is recursive in } X^{(\beta)}]\}.$$
3. *M is equivalent to the map $X \mapsto 2^{\mathbb{N}}$.*

In items 1 and 2, we fix a presentation of α and define the relevant iterations of the jump for all X relative to which that presentation is recursive.

Thus, the only Borel closure operators are those obtained by iterating the Turing jump. We will prove Theorem 3.1 in Section 3. We note that both the theorem and its proof have precursors in Slaman and Steel's [1989] analysis of the Borel order preserving functions from $\mathcal{D}$ to $\mathcal{D}$.

It follows from Theorem 3.1 that the Borel closure operators are well-ordered. In analogy to Martin's conjectures concerning degree invariant functions, one can ask whether the Axiom of Determinacy implies that the set of all closure operators is well-ordered. The fact that we will use Martin's [1975] theorem that all Borel games are determined to prove Theorem 3.1 could be taken as evidence supporting an affirmative answer. On the other hand, this use of Borel determinacy may not be necessary. It is not known whether the proof of Theorem 3.1, like Borel Determinacy, requires the use of uncountably many iterations of the power set operation.

1.4 A Degree Theoretic Definition of the Jump

For our second application, we outline Shore and Slaman's [1999] proof that the jump is definable by a first order formula in $\mathcal{D}$, the partial order of the Turing degrees.

First, we invoke a theorem of Slaman and Woodin.

Theorem 1.11 (Slaman and Woodin [n.d.]) *The function $x \mapsto x''$ is first order definable in $\mathcal{D}$.*

Theorem 1.11 is proved by applying the Slaman and Woodin analysis of automorphisms of the Turing degrees. This analysis involves a fair amount of metamathematics, but not much technical recursion theory.

Second, we apply Theorem 2.1 to the canonical 2-*REA* operator, the double-jump. And so, we define $0'$ in $\mathcal{D}$.

Theorem 1.12 (Shore and Slaman [1999]) *$0'$ is defined within $\mathcal{D}$ as the greatest degree z such that there is no g such that $z \vee g$ is equal to g''.*

Proof: Clearly no degree less than or equal to $0'$ can join any g to g''. Theorem 2.1 states that any degree not below $0'$ does join some g up to g''. Consequently, $0'$ is first order definable in terms of order, join and the double-jump. Order and join are clearly definable in $\mathcal{D}$ and, by Theorem 1.11, so is the double-jump. ∎

We can relativize the above proof to obtain the following definition of the function mapping x to x' in $\mathcal{D}$.

Theorem 1.13 (Shore and Slaman [1999]) *For any degree x, x' is definable from x within $\mathcal{D}$ as the greatest degree z such that there is no g greater than or equal to x such that $z \vee g$ is equal to g''.*

We note that Cooper has claimed to define the jump by different means.

1.5 Recursive enumerability

Finally, we investigate whether the above properties of the jump have analogs for the relation of relative recursive enumerability.

To set the context, recall that recursive enumerability generates a finer hierarchy of degrees than the hyperarithmetic hierarchy.

Definition 1.14 (Ershov, 1968) The *Difference Hierarchy*:

1. A set X is n-recursively enumerable if it has a recursive approximation
$$X(k) = \lim_{s\to\infty} \psi(k,s)$$
such that there are at most n many numbers s such that $\psi(k, s+1) \neq \psi(k, s)$.

2. For infinite α, X is α-recursively enumerable relative to a recursive system S of notations for ordinals if and only if there is a partial recursive function ψ such that for all k, $X(k)$ is equal to $\psi(k, z)$, where z is the S-least notation b for an ordinal less than α such that $\psi(k, b)$ converges.

Jockusch and Shore [1984] showed that every α-re set is α-REA. The converse fails, as every α-re set is Δ^0_2.

In fact, every Δ^0_2 set appears in the difference hierarchy, in the following precise sense.

Theorem 1.15 (Ershov, 1968) *For every path S through $\mathcal{O}$, Kleene's complete set of notations for the recursive ordinals, and for every set $X \in \Delta^0_2$, there is a recursive ordinal α such that X is α-recursively enumerable relative to S.*

However, there is no analog for Spector's theorem. The stratification of the Δ^0_2 sets by the difference hierarchy depends on S, the choice of notations for the recursive ordinals.

1.5.1 Inversion and join

How do inversion and join apply to the operators which appear in the difference hierarchy? Since the difference hierarchy is contained the the REA-hierarchy, Theorem 1.7 applies to increasing α-re operators and provides an inversion theorem for them. We also suspect it provides the optimal result, although that is yet to be proven. The following example shows that Theorem 2.1 cannot be improved to reflect the resolution of the difference hierarchy.

Theorem 4.1 *There is a 2-re operator $D : Z \mapsto D(Z) \geq_T Z$ and a 2-re set X with the following properties.*

1. *X is not of recursively enumerable degree.*
2. *For all $G \subseteq \mathbb{N}$, $X \oplus G \not\equiv_T D(G)$.*

We will present the proof of Theorem 4.1 in Section 4.

The existence of a counterexample to a sharp join theorem would lead us to believe that recursive enumerability and the difference hierarchy behave differently than the jump and the hyperarithmetic hierarchy. The following questions test this hypothesis.

A. Is there an implicit characterization for recursive enumerability and/or for the difference hierarchy?

B. Is there a degree theoretic definition of the relation *y is recursively enumerable relative to x*?

Of course, the second question is well known. See [Rogers, 1967].

1.5.2 Σ-closure operators

In the following, we are again thinking of $M(X)$ as the set of $Y \in 2^{\mathbb{N}}$ which are definable from X, but we are not assuming that the degrees of the definable sets are closed downward in $\mathcal{D}$.

Definition 1.16 A Σ-*closure operator* is a map $M : 2^{\mathbb{N}} \to 2^{2^{\mathbb{N}}}$ with the following properties.

1. For all $X \in 2^{\mathbb{N}}$, $X \in M(X)$.
2. For all $X \in 2^{\mathbb{N}}$ and all $\{Z_1, \dots, Z_k\} \subseteq M(X)$, every set which is Turing equivalent to the join $\bigoplus_{i \leq k} Z_i$ is an element of $M(X)$.
3. For all X and Y in $2^{\mathbb{N}}$, if X is recursive in Y then $M(X) \subseteq M(Y)$.

Every closure operator is a Σ-closure operator, but there are others as well. For example, the maps sending X to the collection of sets with degree recursively enumerable in X or to the collection of sets with degree 2-re in X are Σ-closure operators which are not closure operators.

An implicit characterization of *REA*. The relation $Y \in REA(X)$ is an invariant of the class of Σ-closure operators.

Theorem 5.2 *For any Borel Σ-closure operator M, if there is a cone of X's for which $M(X) \not\subseteq \Delta^0_1(X)$, then there is a cone of X's such that $M(X)$ contains all of the sets which are REA in X.*

In the sense of Theorem 5.2, recursive enumerability is an unavoidable consequence of nontriviality. Additionally, Theorem 5.2 provides operational limits on the possible uniformly definable in X divisions of the class $REA(X)$. We prove it in Section 5.

Despite Theorem 5.2, there is no classification of the Borel Σ-closure operators as simply presented as the one for closure operators given in Theorem 3.1. Shore has pointed out that the *REA* and difference hierarchies give different resolutions of Δ^0_2, so there are natural incomparable (under eventual pointwise inclusion) Σ-closure operators. Horowitz has shown that the Borel Σ-closure operators are not well-founded.

1.5.3 Is relative recursive enumerability degree theoretically definable?

Cooper has claimed that the relation *w is recursively enumerable relative to x* is definable in $\mathcal{D}$, but his proof relied on a join principle for 2-re operators which is contradicted by Theorem 4.1. Cooper has since claimed to use additional properties of specific 2-re operators to circumvent this problem, but the full proof is not yet available.

2 Proving the join theorem

In this section, we outline the proof of the Shore and Slaman [1999] Join Theorem for α-*REA* operators.

Theorem 2.1 (Shore and Slaman [1999]) *Suppose that J is an α-REA operator and for all $\beta < \alpha$, $\emptyset^{(\beta)} \not\geq_T X$. There is a G such that $X \oplus G \equiv_T J(G)$.*

We first show that it is sufficient to prove a weak join theorem for the α-*REA* operators $X \mapsto X^{(\alpha)}$. Namely, it is sufficient to show that if X and α are given so that α is a recursive ordinal and for all $\beta < \alpha$, $\emptyset^{(\beta)} \not\geq_T X$, then there is a G such that $X \oplus G \geq_T G^{(\alpha)}$.

Suppose that X and α are given as above and that J is an α-*REA* operator. Assuming the weak join theorem for the α-jump, fix G so that $X \oplus G \geq_T G^{(\alpha)}$. By Theorem 1.7 relative to G, there is an $H \geq_T G$ such that $J(H) \equiv X \oplus G$. But then $X \oplus H \geq_T X \oplus G \equiv_T J(H)$. Similarly, $J(H) \equiv_T X \oplus G \geq_T X$, since J is increasing in degree $J(H) \geq_T H$, and so $J(H) \geq_T X \oplus H$. Thus, $J(H) \equiv_T X \oplus H$, as required to verify the join theorem for J.

Turing functionals. We begin with a formalization of the way by which one set is recursive in another.

Definition 2.2 1. A *Turing functional* Φ is a set of sequences (x, y, σ) such that x and y are natural numbers and σ is a finite binary sequence. Further, for all x, for all y_1 and y_2, and for all compatible σ_1 and σ_2, if $(x, y_1, \sigma_1) \in \Phi$ and $(x, y_2, \sigma_2) \in \Phi$, then $y_1 = y_2$ and $\sigma_1 = \sigma_2$.

2. Φ is *use-monotone* if the following conditions hold.

   (a) For all (x_1, y_1, σ_1) and (x_2, y_2, σ_2) in Φ, if σ_1 is a proper initial segment of σ_2, then x_1 is less than x_2.

   (b) For all x_1 and x_2, y_2 and σ_2, if $x_2 > x_1$ and $(x_2, y_2, \sigma_2) \in \Phi$, then there are y_1 and σ_1 such that $\sigma_1 \subseteq \sigma_2$ and $(x_1, y_1, \sigma_1) \in \Phi$.

3. We write $\Phi(x, \sigma) = y$ to indicate that there is a τ such that τ is an initial segment of σ, possibly equal to σ, and $(x, y, \tau) \in \Phi$. If $X \subseteq \mathbb{N}$, we write $\Phi(x, X) = y$ to indicate that there is an ℓ such that $\Phi(x, X \upharpoonright \ell) = y$, and write $\Phi(X)$ for the function evaluated in this way. We say that $\Phi(X) = Y$ if and only if $\Phi(X)$ is equal to the characteristic function of Y.

In Definition 2.2, we do not require that Φ be recursively enumerable. Consequently, if Φ is a Turing functional and $X \subseteq \mathbb{N}$, then $\Phi(X)$ is recursive only in the join of Φ and X. Note also that in this formulation, a Turing functional is just a particular way to define a continuous function from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$.

In this language, $X \geq_T Y$ if and only if there is a recursive Turing functional Φ such that $\Phi(X) = Y$.

Kumabe–Slaman forcing. Kumabe and Slaman introduced the following notion of forcing in an earlier join theorem for the ω-jump.

1. A *condition* is a pair $p = (\Phi_p, \boldsymbol{Z}_p)$ consisting of a finite use-monotone Turing functional and a finite subset of $2^{\mathbb{N}}$.
2. If $p = (\Phi_p, \boldsymbol{Z}_p)$ and $q = (\Phi_q, \boldsymbol{Z}_q)$ are conditions, then $p \geq q$ if and only if
   (a) i. $\Phi_p \subseteq \Phi_q$ and
       ii. for all $(x_q, y_q, \sigma_q) \in \Phi_q \setminus \Phi_p$ and all $(x_p, y_p, \sigma_p) \in \Phi_p$, the length of σ_q is greater than the length σ_p,
   (b) $\boldsymbol{Z}_p \subseteq \boldsymbol{Z}_q$,
   (c) for every x, y, and $Z \in \boldsymbol{Z}_p$, if $\Phi_q(x, Z) = y$ then $\Phi_p(x, Z) = y$.

Hence, a condition p specifies all of the elements (x, y, σ) in Φ for which σ has length less than or equal to the maximum length occurring in Φ_p and specifies finitely many sets relative to which Φ is only defined at arguments where Φ_p is already defined.

2.1 An illuminating special case

By fixing a recursive α, it is sufficient to build a Φ with the following properties.

1. Every atomic statement about $\Phi^{(\alpha)}$ is decided by a Kumabe-Slaman condition on Φ.
2. $\Phi(X) = \Phi^{(\alpha)}$. Hence, $X \oplus \Phi \geq_T \Phi^{(\alpha)}$.

We will only give the argument for the case when $\alpha = 1$. The general proof involves a transfinite analysis of the Kumabe-Slaman forcing relation for atomic statements about $\Phi^{(\alpha)}$.

Deciding $\Sigma^0_1(\Phi)$ sentences. Our goal is to decide an atomic statement about Φ' while maintaining $\Phi(X) = \Phi'$.

Given a condition $p = (\Phi_p; \boldsymbol{Z}_p)$ and a Σ^0_1-sentence $\psi(\Phi)$, one of the following conditions holds.

1. There is a $q = (\Phi_q, \boldsymbol{Z}_p)$ extending p such that $\Phi_p(X) = \Phi_q(X)$, and $\psi(\Phi_q)$ is satisfied by means of a witness less than the length of some σ such that there are x and y with $(x, y, \sigma) \in \Phi_q$.
2. For all q extending p, if $\psi(\Phi_q)$ then there is a $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ such that X extends σ.

In the first case, $(\Phi_q, \boldsymbol{Z}_p)$ is a condition extending p as required.

Otherwise, it is not possible to extend Φ_p to make ψ hold without adding a computation relative to X or one of the elements of $\boldsymbol{Z}_p$. We convert this situation into a definition.

Definition 2.3 Fixing k, $\tau \in (2^{k_1})^k$ is *essential* if and only if for all finite Φ_q extending Φ_p, if $\psi(\Phi_q)$, then there is a $(x, y, \sigma) \in \Phi_q \setminus \Phi_p$ such that σ is compatible with some component of τ.

The set of essential k-sequences form a finitely branching Π^0_1 tree T.

Now consider the second case in the analysis of ψ. Let k be the size of $\boldsymbol{Z}_p \cup \{X\}$. $\boldsymbol{Z}_p \cup \{X\}$ determines an infinite path through T.

Since X is not recursive, T has another path with coordinates $\boldsymbol{Y}$ such that $X \notin \boldsymbol{Y}$. Consequently, $q = (\Phi_p, \boldsymbol{Z}_p \cup \boldsymbol{Y})$ forces $\psi(\Phi)$ and is the desired condition.

3 Proof of the hierarchy theorem

Theorem 3.1 *If M is a closure operator such that the relation $Y \in M(X)$ is Borel, then one of the following conditions holds.*

1. *There is a countable ordinal α such that M is equivalent to the map*

$$X \mapsto \{Y : Y \textit{ is recursive in } X^{(\alpha)}\}.$$

2. *There is a countable ordinal α such that M is equivalent to the map*

$$X \mapsto \{Y : (\exists \beta < \alpha)[Y \textit{ is recursive in } X^{(\beta)}]\}.$$

3. *M is equivalent to the map $X \mapsto 2^{\mathbb{N}}$.*

In items 1 and 2, we fix a presentation of α and define the relevant iterations of the jump for all X relative to which that presentation is recursive.

Proof: In this argument, we apply some effective descriptive set theory. The reader may consult [Sacks, 1990] for the relevant background material.

First, suppose that there is a cone of X's such that $M(X)$ does not include all of the sets which are hyperarithmetic in X. Fix B so that B is the base of such a cone and so that the relation $Y \in M(X)$ is Δ^1_1 relative to B. Then, for all $X \geq_T B$, there is an $\alpha(X)$ such that $\omega_1^X > \alpha(X)$ and $X^{(\alpha(X))} \notin M(X)$. By Spector's [1955] Bounding Theorem, there is a single α such that $\omega_1^B > \alpha$ and for all $X \geq_T B$, $X^{(\alpha)} \notin M(X)$. Let α_0 be such an α.

By Martin's [1975] Borel Determinacy, for every β less than α_0, either there is a cone of degrees X such that $X^{(\beta)} \in M(X)$ or there is a cone of degrees X such that $X^{(\beta)} \notin M(X)$. Since every countable set of degrees has an upper bound and α_0 is countable, by increasing B and decreasing α_0 as needed, we may assume that for all β less than α_0 and all $X \geq_T B$, $X^{(\beta)} \in M(X)$ and that for all $X \geq_T B$, $X^{(\alpha_0)} \notin M(X)$.

Now, we claim that for all $X \geq_T B$, if $Y \in M(X)$ then there is a $\beta < \alpha_0$ such that $Y \leq_T X^{\beta}$. Suppose not, and let X and Y be a counterexample to the claim. By Theorem 2.1 relative to X, there is a G such that $G \geq_T X$ and $Y \oplus G \geq_T G^{(\alpha_0)}$. But then, since $G \geq_T X$, $M(X) \subseteq M(G)$ and so $Y \in M(G)$.

Since M is a closure operator, $G \in M(G)$ and so $Y \oplus G \in M(G)$. Consequently, $G^{(\alpha_0)} \in M(G)$, which is a contradiction.

Thus for all $X \geq_T B$ and all Y, $Y \in M(X)$ if and only if there is a $\beta < \alpha_0$ such that Y is recursive in $X^{(\alpha_0)}$, and the theorem is proven for this case.

Now, suppose that in every cone there is an X such that $M(X)$ includes all of the sets which are hyperarithmetic in X.

Fix B so that the relation $Y \in M(X)$ is Δ^1_1 relative to B and so that every set hyperarithmetic in B belongs to $M(B)$. Let $HYP(B)$ denote the collection of sets Y such that Y is hyperarithmetic in B. Since $HYP(B)$ is not Δ^1_1 relative to B, let Y be a set in $M(B) \setminus HYP(B)$. By a result of Woodin (unpublished), a join theorem for the hyperjump, there is a $G \geq_T B$ such that $Y \oplus G \equiv_T \mathcal{O}^G$, where $\mathcal{O}^G$ is the complete Π^1_1 subset of $\mathbb{N}$ relative to G. As above, $\mathcal{O}^G$ is an element of $M(G)$. But now consider the set $\{Y : Y \notin M(G)\}$. This set is Δ^1_1 relative to G. If it were nonempty, then it would have an element recursive in $\mathcal{O}^G$. But $\mathcal{O}^G \in M(G)$ implies that every set recursive in $\mathcal{O}^G$ belongs to $M(G)$. Consequently, $M(G) = 2^{\mathbb{N}}$. Finally, note that if $X \geq_T G$ then $M(G) \subseteq M(X)$. Thus, for every X, if $X \geq_T G$ then $M(X) = 2^{\mathbb{N}}$, and the theorem is proven in this case as well. ■

4 A 2-re operator without the join property

Theorem 4.1 *There is a 2-re operator $D : A \mapsto D(A) \oplus A$ and a set X such that the following conditions hold.*

1. *The Turing degree of X is not recursively enumerable.*
2. *For all A, $D(A) \oplus A$ and $X \oplus A$ have different Turing degrees.*

We present the proof of Theorem 4.1 in two parts. In Section 4.1, we construct a 2-re set X. As stated above, we will ensure that the Turing degree of X is not recursively enumerable. We will also ensure that the recursive approximation to X is *self-restraining*, a dynamic feature which we explain in Definition 4.3. In Section 4.2, we start with any Δ^0_2 set X with a self-restraining recursive approximation, and we produce a 2-re increasing operator D such that for all A, $D(A)$ and $X \oplus A$ do not have the same Turing degree. Theorem 4.1 follows: apply the method of Section 4.2 to the set of Section 4.1 to produce D and X as required.

Definition 4.2 For Φ a Turing functional, let φ be the functional such that $\varphi(x, X) = \ell$ if and only if $(x, y, X \upharpoonright \ell) \in \Phi$.

In other words, $\varphi(x, X)$ is the amount of X used to determine the value of $\Phi(x, X)$. Note that $\Phi(X)$ and $\varphi(X)$ have the same domain.

We will assume that all recursive Turing functionals are use-monotone; see Definition 2.2. In particular, if Φ is a recursive Turing functional then, by our convention, for every X, $\varphi(X)$ is a nondecreasing function. We do not lose any generality: if Y is recursive in X, then there is a use-monotone recursive Φ such that $\Phi(X) = Y$.

4.1 Constructing X

In the following, we will write $X(n)[s]$ to denote the value of a recursive binary predicate at the pair of arguments n and s. We say that this predicate approximates a set if for all n, $\lim_{s\to\infty} X(n)[s]$ exists and is equal to 0 or 1. In this case, we will write $\lim_{s\to\infty} X(n)[s] = X(n)$. We are anticipating the case in which $X(n)[s]$ is the approximation to $X(n)$ during stage s.

We will also approximate the application of a recursive functional Φ to sets which are also being approximated. We will use the suffix $[s]$ to indicate the approximation to the preceding expression during stage s. We adopt the usual convention that we will only approximate a functional's having a value at stage s when the use of that functional is less than s.

Definition 4.3 Suppose that $\lim_{s\to\infty} X(n)[s] = X(n)$. We say that $X(n)[s]$ is a *self-restraining* approximation to X if and only if there is an increasing recursive function g from $\mathbb{N}$ to $\mathbb{N}$ such that for all ℓ and all s, if $X(\ell)[s] \neq X(\ell)[s+1]$ then there are less than $g(\ell)$ numbers $t > s$ such that $(\exists m \leq s)\,[X(m)[t] \neq X(m)[t+1]]$.

The following construction is not original to this paper (cf. [Jockusch and Shore, 1984, Theorem 1.6]). We reproduce it here so that we can verify that the recursive approximation to the set constructed is self-restraining.

Theorem 4.4 *There is recursive approximation $X(n)[s]$ to a 2-re set X such that the following conditions hold.*

1. *The Turing degree of X is not recursively enumerable.*
2. *The approximation $X(n)[s]$ is self-restraining.*

Proof: We define $X(n)[s]$ by recursion on s, in the context of a finite injury construction. We begin by setting $X(n)[0]$ equal to 0. Equivalently, during stage 0, X is empty. In a later stage $s+1$, we may add n to X by setting $X(n)[s+1]$ equal to 1 when $X(n)[s]$ was equal to 0, or we may remove n from X by setting $X(n)[s+1]$ equal to 0 when $X(n)[s]$ was equal to 1. For each n, we will add n to X at most once and remove n from X at most once, and so we will construct a 2-re set.

A single requirement. Suppose that W is a recursively enumerable set and that Φ and Ψ are Turing functionals. We must satisfy the following requirement.

$$\Psi(W) \neq X \text{ or } \Phi(X) \neq W$$

Our strategy works as follows.

1. Choose a number n larger than the current stage and larger than any number ever mentioned in the construction prior to this point. Prohibit n from entering X until reaching a stage s during which $\Psi(n, W)[s] = 0$,

predicting that n is not an element of X, and for all m less than or equal to $\psi(n,W)[s]$, $\Phi(m,X)[s]$ is equal to $W(m)[s]$. We can visualize this situation as in Figure 1.

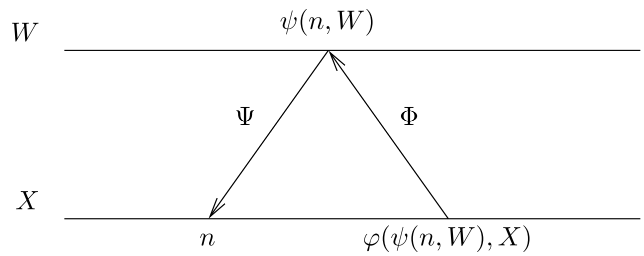

Fig. 1: Configuring $\Phi(W) \neq X$ or $\Phi(X) \neq W$.

Upon reaching such a stage s_1, go to Step 2.

2. Put n into X upon entering Step 2 in stage s_1.

   Wait for a later stage $s > s_1$ such that $\Psi(n,W)[s] = 1$. While waiting for that stage, prohibit any strategy of lower priority from changing X at any number less than or equal to s_1. Upon reaching such a stage s_2, go to Step 3.

3. Take n out of X, and prohibit any strategy of lower priority from changing X at any number less than or equal to s_2. Thereby, we set $X \upharpoonright s_1$ equal to $X[s_1] \upharpoonright s_1$ (as approximated before we put n into X).

The strategy has three possible outcomes. It could wait forever in one of the first two steps, in which case the requirement is clearly satisfied. Or the strategy could reach Step 3. In this last case, X is equal to $X[s_1]$ on all numbers less than or equal to s_1. But some number m less than $\psi(n,W)[s_1]$ must have entered W between stages s_1 and s_2 as $\Psi(n,W)[s_1] = 0$ and $\Psi(n,W)[s_2] = 1$. Consequently, if the strategy reaches Step 3, then for this m, $\Phi(m,X)$ is not equal to $W(m)$.

Priority construction. We define the recursive approximation $X(n)[s]$ to X by applying the finite injury priority method. We arrange the strategies in order type ω, with strategies later in the list having lower priority. During stage s, we identify the highest priority strategy which is supposed to take action, either by choosing its value for n or by going from one step to another, and follow the instructions for that strategy. We say that the lower priority strategies are injured, and we return them to the state in which they will begin Step 1 during the next stage.

Since each strategy acts only finitely often and for each strategy there are only finitely many others of higher priority, each strategy will eventually be implemented without injury and will satisfy its associated requirement.

Self-restraint. Suppose that at stage s we change our approximation to X at ℓ. Since only the first ℓ strategies can change the approximation to X at ℓ, each strategy of index greater than or equal to ℓ is injured during stage s and will not change X below s at any later stage. Similarly, each strategy of index less than ℓ can change our approximation to X at most twice below s before it is injured and then unable to change X below s. Consequently, we change X at a number less than or equal to s at most 2ℓ many times after stage s. It follows that our construction is self-restraining.

This ends the proof of Theorem 4.4. ∎

4.2 Constructing D

Suppose that X is a Δ_2^0 set which is not of recursively enumerable degree and has a self-restraining approximation $X(n)[s]$. It is safe to think of X as being the set constructed during the proof of Theorem 4.4.

We construct the required 2-re operator. In our presentation, we will assume that we are given the set A and will uniformly describe the set $D(A)$ so that it is 2-re relative to A.

We construct $D(A)$ by a finite injury construction similar to that in the previous section.

4.2.1 A single requirement

Suppose that Φ and Ψ are Turing functionals. We must satisfy the following requirement.

$$\Psi(X \oplus A) \neq D(A) \text{ or } \Phi(D(A) \oplus A) \neq X$$

We fix a recursive partition of $\mathbb{N}$ into infinitely many infinite subsets, and we allocate one of these subsets, R, to this requirement. In the following description of our strategy, the stage s approximation $D(A)[s]$ to $D(A)$ refers to the state of $D(A)$ at the beginning of stage s in the recursion being defined relative to A. In contrast, the stage s approximation $X[s]$ to X refers to the self-restraining recursive approximation which is given.

1. Choose a number $m_0 \in R$ larger than the current stage and larger than any number ever mentioned in the construction prior to this point. Prohibit any element of R which is greater than or equal to m_0 from entering $D(A)$ until reaching a stage s during which the following conditions hold.

   - $\Psi(m_0, X \oplus A)[s] = 0$, predicting that m_0 is not an element of $D(A)$. Let $\ell = \psi(m_0, X \oplus A)[s]$ be the length of the associated computation.
   - For each i less than or equal to $g(\ell)+1$, $\Psi(m_i, X \oplus A)[s] = 0$, where $m_1, \ldots, m_{g(\ell)+1}$ are the first $g(\ell)+1$ elements of R which are strictly greater than m_0. Let ℓ^* be $\psi(m_{g(\ell)+1}, X \oplus A)[s]$. By Convention 4.2, ℓ^* is the least upper bound of $\psi(m_i, X \oplus A)[s]$ for i between 1 and $g(\ell)+1$.

- For all m less than or equal to ℓ^*, $\Phi(m, D(A) \oplus A)[s]$ is equal to $X(m)[s]$.

We can visualize this situation as in Figure 2.

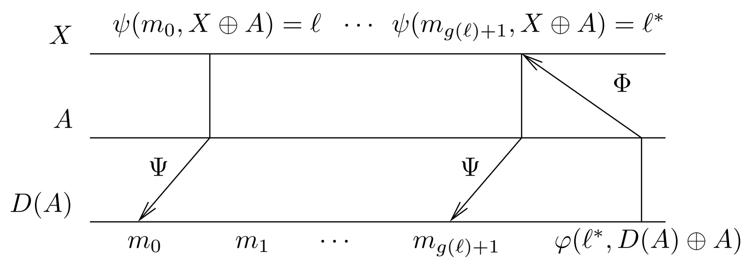

Fig. 2: Configuring $\Psi(X \oplus A) \neq D(A)$ or $\Phi(D(A) \oplus A) \neq X$.

Upon reaching such a stage s_0, let $j = 0$ and go to Step 2a.

2. (a) Put m_j into $D(A)$ upon entering Step 2a.
   Wait for a later stage $t > s_j$ such that $\Psi(m_j, X \oplus A)[t] = 1$. While waiting for that stage, prohibit any strategy of lower priority from changing $D(A)$ at any number less than or equal to s_0, thus we maintain the viability of the computations seen during stage s_0. Upon reaching such a stage t_j, go to Step 2b.

   (b) Remove m_j from $D(A)$ upon entering Step 2b, thereby returning the value of $D(A) \upharpoonright s_0$ to $D(A)[s_0] \upharpoonright s_0$.
   Wait for a later stage $s > t_j$ such that $\Phi(m_j, D(A) \oplus A)[s] = X[s]$ on all numbers less than or equal to ℓ^*. While waiting for that stage, prohibit any strategy of lower priority from changing $D(A)$ at any number less than or equal to s_0. Upon reaching such a stage s, increase the value of j by 1, define s_j to be equal to s, and go to Step 2a with these values for j and s_j.

This strategy could wait forever in the Step 1, in which case the requirement is clearly satisfied. Once the strategy leaves Step 1, it has defined ℓ and has $2(g(\ell) + 1)$ possible outcomes. If for some fixed $j_0 \leq g(\ell) + 1$, it waits forever in Step 2a with $j = j_0$ then the requirement is again clearly satisfied. Similarly, the requirement is satisfied if for some fixed $j_0 < g(\ell) + 1$, it waits forever in Step 2b with $j = j_0$.

The last possibility is for the strategy to reach Step 2b with $j = g(\ell) + 1$. In particular, it went from Step 2a to Step 2b during some stage t_0 after s_0. But then, $\Psi(m_0, X \oplus A)[s_0] = 0$ and $\Psi(m_0, X \oplus A)[t_0] = 1$. It can only be that the approximation to X changed during some stage s between s_0 and t_0 at some number ℓ_0 less than or equal to $\psi(m_0, X[s] \oplus A)[s_0] = \ell$. By the assumption that the approximation to X is self-restraining, the approximation to X changes no more than $g(\ell_0)$ times at numbers less than or equal to s during stages after s. Now we can check some inequalities. By the terms of Definition 4.3 g is increasing, and so the approximation to X changes no more than $g(\ell)$ times at numbers less than or equal to s during stages after s. Since $\psi(m_{g(\ell)+1}, X[s_0] \oplus A) = \ell^*$ is defined at stage s_0, it's value is less than s_0. Consequently, the approximation to $X \upharpoonright \ell^*$ changes no more than $g(\ell)$ times during stages after s. Finally, s is less than t_0, and so the approximation to $X \upharpoonright \ell^*$ changes changes no more than $g(\ell)$ times during stages after t_0.

But each time the strategy went from one step to the next, it must be that the approximation to $X \upharpoonright \ell^*$ changed between the stage when the strategy entered that step and the one when it went to the next step. Either, in reaction to our adding m_j to $D(A)$, the approximation to $X \upharpoonright \ell^*$ changed to make $\Psi(m_j, X \oplus A)[s] = 1$, and therefore it became incompatible with $X[s_0] \upharpoonright \ell^*$. Or, in reaction to our returning the value of $D(A) \upharpoonright \ell^*$ to $D(A) \upharpoonright \ell^*[s_0]$, the approximation to X below ℓ^* returned to $X[s_0] \upharpoonright \ell^*$ in order to agree with $\Phi(m_j, D(A) \oplus A)[s_0] \upharpoonright \ell^*$. But then the approximation to $X \upharpoonright \ell^*$ changed at least $2(g(\ell)+1)-1 = 2g(\ell)+1$ times after stage t_0. Notice that $g(\ell) < 2g(\ell)+1$, and that we have a contradiction.

It follows that the strategy cannot reach Step 2b for $j = \ell + 1$ and that the requirement is satisfied.

4.2.2 Priority Construction

As in the previous section, we organize our strategies in a finite injury priority construction. Once the strategies of higher priority have stopped acting, the next one begins in Step 1 and ensures that its associated requirement is satisfied.

This completes the proof of Theorem 4.1 ∎

5 An implicit characterization of *REA*

In this section, we prove that every nontrivial Σ-closure operator eventually extends the map $X \mapsto REA(X)$. Our proof makes use of the Shoenfield Jump Inversion Theorem.

Theorem 5.1 (Shoenfield [1959]) *Suppose that W is REA relative to $\emptyset'$. Then there is a set W_0 such that $\emptyset' \geq_T W_0$ and $W_0' \equiv_T W$.*

Theorem 5.2 *For any Borel Σ-closure operator M, if there is a cone of X's for which $M(X) \not\subseteq \Delta^0_1(X)$, then there is a cone of X's such that $M(X)$ contains all of the sets which are REA in X.*

Proof: Suppose that M is a Borel Σ-closure operator and that there is a cone of X's for which $M(X) \not\subseteq \Delta^0_1(X)$. Choose B_1 so that for all X, if $X \geq_T B_1$ then $M(X) \not\subseteq \Delta^0_1(X)$.

We first apply the argument from Section 3 to conclude the there is a cone of X's for which $X' \in M(X)$. Suppose that $X \geq_T B_1$. Let A be an element of

$M(X)$ such that $X \not\geq_T A$. By Theorem 1.3 relativized to X, choose G so that $G \geq_T X$ and $A \oplus G \equiv_T G'$. Then, $A \in M(X) \subseteq M(G)$ and $G \in M(G)$, and so $G' \in M(G)$, since it is equivalent to the join of elements of $M(G)$.

Consequently, for every X, there is a G such that $G \geq_T X$ and $G' \in M(G)$. Martin's [1975] Borel Determinacy implies that there is a cone of G's such that $G' \in M(G)$. Choose B_2 so that for every G, if $G \geq_T B_2$ then $G' \in M(G)$.

We will now argue that B_2' is the base of a cone as required. Suppose that H is given with $H \geq_T B_2'$. Let W be a set which is REA relative to H. By Theorem 1.2 relativized to B_2, fix H_0 so that $H_0 \geq_T B_2$ and $H_0' \equiv_T H$. Finally, by Theorem 5.1 relativized to H_0, choose W_0 so that $H \geq_T W_0 \geq_T H_0$ and $W_0' \equiv_T W$. In Figure 5, we indicate the relationships between the degrees of these sets. We use solid lines to indicate the Turing order with the higher set being $\geq_T$ the lower one, and we use dotted lines to indicate the Turing jump with the higher set having the same Turing degree as the jump of the lower one.

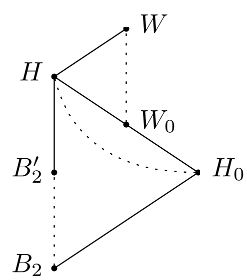

Fig. 3: Relationships between B_2, B_2', the H's, and the W's.

Now, $W_0 \geq_T B_2$ implies that $W_0' \in M(W_0)$. Since $H \geq_T W_0$, $M(W_0) \subseteq M(H)$. Consequently, $W_0' \in M(H)$. Since $W_0' \equiv_T W$, $W \in M(H)$. Since W was an arbitrary set REA relative to H, every set REA relative to H belongs to $M(H)$. Since H was an arbitrary element of the cone above B_2', for every set H in the cone above B_2', all of the sets which are REA relative to H belong to $M(H)$, as required. ■

It is open whether there is a Borel Σ-closure operator M with the following property: there is a cone of X's, such that

1. there is a set in $M(X)$ whose degree is not recursively enumerable relative to X
2. and there is a set $D \geq_T X$ which is 2-re in X and not in $M(X)$.

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