Σ_n-Bounding and Δ_n-Induction

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Abstract

Working in the base theory of $\mathrm{PA}^- + \mathrm{I}\Sigma_0 + \exp$, we show that for all $n \in \omega$, the bounding principle for Σ_n-formulas ($\mathrm{B}\Sigma_n$) is equivalent to the induction principle for Δ_n-formulas ($\mathrm{I}\Delta_n$). This partially answers a question of J. Paris; see Clote and Krajíček (1993).

1 Introduction

We begin with some background material on first order arithmetic. However, in lieu of giving a detailed introduction to the subject, we settle for recommending the excellent texts Kaye (1991) and Hájek and Pudlák (1998).

The language of first order arithmetic consists of the usual symbols of first order logic $\forall$, $\exists$,(,), $\neg$, $\wedge$, $\vee$, $\rightarrow$,$\leftrightarrow$, $=$, and variables $x_1, x_2, \ldots$ together with symbols from arithmetic: 0, 1, $+$,$\cdot$, and $<$. Formulas and sentences are constructed as usual.

To fix some typographical notation, we use $x \leq y$ to indicate $x < y \vee x = y$. We also use boldface characters, such as $\boldsymbol{p}$ and $\boldsymbol{x}$, represent sequences of type p or x. The inequality $\boldsymbol{p} < r$ indicates that every element of the sequence $\boldsymbol{p}$ is less than r.

PA^- consists of the axioms for the nonnegative part of a discretely ordered ring. It is

[*]During the preparation of this paper, Slaman was partially supported by the Alexander von Humboldt Foundation and by National Science Foundation Grant DMS-9988644. Slaman is grateful to Jan Krajíček for reading a preliminary version of this paper and suggesting improvements to it.

the universal closures of the following formulas.

$$
\begin{array}{ll}
x + y = y + x, & x \cdot y = y \cdot x, \\
(x + y) + z = x + (y + z), & (x \cdot y) \cdot z = x \cdot (y \cdot z), \\
x \cdot (y + z) = x \cdot y + x \cdot z, & \\
x + 0 = x, & x \cdot 0 = 0, \\
x \cdot 1 = x, & \\
\neg(x < x), & (x < y \wedge y < z) \to x < z, \\
x < y \vee y < z \vee x = y, & \\
(x < y) \to (x + z < y + z), & (0 < z \wedge x < y) \to (x \cdot z < y \cdot z), \\
x < y \to (\exists z)(x + z = y), & \\
0 < 1, & \\
0 \leq x, & 0 < x \to 1 \leq x
\end{array}
$$

1.1 Bounding, Least Number, and Induction Principles for Σ_n and Π_n-formulas

Bounding for Σ_n-formulas ($\mathrm{B}\Sigma_n$). $\mathrm{B}\Sigma_n$ consists of all sentences

$$(\forall \boldsymbol{p})(\forall a)[(\forall x < a)(\exists \boldsymbol{y})\varphi(x, \boldsymbol{y}, \boldsymbol{p}) \to (\exists b)(\forall x < a)(\exists \boldsymbol{y} < b)\varphi(x, \boldsymbol{y}, \boldsymbol{p})]$$

in which φ is Σ_n. That is, if for every number x less than a, there are numbers $\boldsymbol{y}$ satisfying a Σ_n-property relative to x and parameters $\boldsymbol{p}$, then there is a bound b such that for each x less than a, there is such a $\boldsymbol{y}$ all the elements of which are less than b.

Least number principle for Σ_n-formulas ($\mathrm{L}\Sigma_n$). $\mathrm{L}\Sigma_n$ consists of all the sentences

$$(\forall \boldsymbol{p})\,[(\exists x)(\varphi(x, \boldsymbol{p}) \to (\exists x)[\varphi(x, \boldsymbol{p}) \wedge (\forall y < x)\neg\varphi(y, \boldsymbol{p})]]$$

in which φ is Σ_n. In other words, if A is defined by a Σ_n-formula relative to parameters and A is not empty, then A has a least element. $\mathrm{L}\Pi_n$ is defined similarly.

Induction for Σ_n-formulas ($\mathrm{I}\Sigma_n$). $\mathrm{I}\Sigma_n$ consists of all the sentences

$$(\forall \boldsymbol{p})[(\varphi(0, \boldsymbol{p}) \wedge (\forall x)(\varphi(x, \boldsymbol{p}) \to \varphi(x + 1, \boldsymbol{p}))) \to (\forall x)\varphi(x, \boldsymbol{p})]$$

in which φ is Σ_n. In other words, if A is defined by a Σ_n-formula relative to parameters, $0 \in A$, and A is closed under the successor function, then every number is in A.

A *cut* in a model $\mathfrak{M}$ is a subset J of $\mathfrak{M}$ such that for every x and y, if $x \in J$ and $y < x$ then $y \in J$. A proper cut is a nonempty cut which does not include all of the elements of $\mathfrak{M}$. A nonprincipal cut is a cut which has no greatest element. One way in which Σ_n-induction could fail in $\mathfrak{M}$ would be for there to be a proper nonprincipal cut which is definable by a Σ_n-formula relative to parameters in $\mathfrak{M}$.

The Kirby and Paris Theorem. Building on Parsons (1970), Kirby and Paris (1977) proved the fundamental theorem connecting bounding, induction, and the least number principles for Σ_n and Π_n-formulas.

Theorem 1.1 (Kirby and Paris) *Work in* $\mathrm{PA}^- + \mathrm{I}\Sigma_0$. *For all* $n \geq 1$,

1. $\mathrm{I}\Sigma_n \iff \mathrm{I}\Pi_n \iff \mathrm{L}\Sigma_n \iff \mathrm{L}\Pi_n$.
2. $\mathrm{B}\Sigma_{n+1} \iff \mathrm{B}\Pi_n$.
3. $\mathrm{I}\Sigma_{n+1} \implies \mathrm{B}\Sigma_{n+1} \implies \mathrm{I}\Sigma_n$, *and the implications are strict.*

Consequently, the bounding principles are strictly interleaved with the equivalent induction and least number principles.

1.2 Δ_n-formulas

We form the principles of $\mathrm{L}\Delta_n$ and $I\Delta_n$ by inserting an hypothesis of equivalence between Σ_n and Π_n-formulas.

The least number principle for Δ_n-formulas ($\mathrm{L}\Delta_n$). $\mathrm{L}\Delta_n$ consists of all the sentences

$$(\forall \boldsymbol{p})\left[\begin{array}{c}(\forall x)(\varphi(x,\boldsymbol{p}) \leftrightarrow \neg\psi(x,\boldsymbol{p})) \rightarrow \\ \big((\exists x)(\varphi(x,\boldsymbol{p}) \rightarrow (\exists x)[\varphi(x,\boldsymbol{p}) \wedge (\forall y < x)\neg\varphi(y,\boldsymbol{p})]\big)\end{array}\right]$$

in which φ and ψ are Σ_n. In other words, if A is defined by a Σ_n-formula and also by a Π_n-formula relative to parameters and A is not empty, then A has a least element.

Theorem 1.2 (Gandy (unpublished), see Hájek and Pudlák (1998)) *If* $n \geq 1$ *and* $\mathfrak{M}$ *is a model of* $\mathrm{PA}^- + \mathrm{I}\Sigma_0$, *then*

$$\mathfrak{M} \models \mathrm{B}\Sigma_n \iff \mathfrak{M} \models \mathrm{L}\Delta_n.$$

Proof: $\mathrm{B}\Sigma_n \implies \mathrm{L}\Delta_n$**.** To sketch the proof, suppose that A is a Δ_n-set and suppose that A has an element less than a. Use $\mathrm{B}\Sigma_n$ to bound the witnesses needed to determine for each x less than a, whether x belongs to A. $\mathrm{B}\Sigma_n$ implies that the Π_{n-1} predicates are closed under bounded existential quantification, so the restriction of A to the numbers less than a can be defined by a Π_{n-1}-formula. Now, $\mathrm{L}\Pi_{n-1}$ is provable from $\mathrm{B}\Sigma_n$, and so there is a least element of A below a.

$\mathrm{L}\Delta_n \implies \mathrm{B}\Sigma_n$**.** We follow the Hájek and Pudlák (1998) account of Gandy's proof. We work in the theory $\mathrm{L}\Delta_n$. Suppose that a, $\boldsymbol{p}$, and a Π_{n-1}-formula φ_0 are given so that $(\forall x < a)(\exists \boldsymbol{y})\varphi_0(x,\boldsymbol{y},\boldsymbol{p})$. We must argue that $(\exists b)(\forall x < a)(\exists \boldsymbol{y} < b)\varphi_0(x,\boldsymbol{y},\boldsymbol{p})$.

Define $\theta(z,a,\boldsymbol{p})$ to be the formula

$$z \leq a \wedge (\exists u)[(\forall x)(z \leq x < a \rightarrow (\exists \boldsymbol{y} < u)\varphi_0(x,\boldsymbol{y},\boldsymbol{p})) \wedge (\forall v < u)(\forall \boldsymbol{y} < v)\neg\varphi_0(z,\boldsymbol{y},\boldsymbol{p})]$$

One can read $\theta(z,a,\boldsymbol{p})$ as saying that for any x with $z \leq x < a$, the least witness to $(\exists \boldsymbol{y})\varphi_0(x,\boldsymbol{y},\boldsymbol{p})$ is less than or equal to the least witness to $(\exists \boldsymbol{y})\varphi_0(z,\boldsymbol{y},\boldsymbol{p})$. Syntactically, $\theta(z,a,\boldsymbol{p})$ is equivalent to a formula of the form $(\exists u)\psi$, where ψ is obtained by the bounded quantification of a Π_{n-1}-formula. Now, $\mathrm{L}\Delta_n \implies \mathrm{I}\Sigma_{n-1}, \mathrm{I}\Sigma_{n-1} \implies \mathrm{B}\Sigma_{n-1}$, and $\mathrm{B}\Sigma_{n-1}$ proves that the Π_{n-1} predicates are closed under bounded quantification. So, we can conclude that $\theta(z, a, \boldsymbol{p})$ is equivalent to a Σ_n-formula. Additionally, if the least witness for z is greater than or equal to some witness for x, then every witness for z is greater than or equal to some witness for x. So, $\theta(z, a, \boldsymbol{p})$ is equivalent to the following formula.

$$z \leq a \wedge (\forall u)[(\exists \boldsymbol{y} < u)\varphi_0(z, \boldsymbol{y}, \boldsymbol{p}) \rightarrow (\forall x)(z \leq x < a \rightarrow (\exists \boldsymbol{y} < u)\varphi_0(x, \boldsymbol{y}, \boldsymbol{p})]$$

Applying $\mathrm{B}\Sigma_{n-1}$ as above, this formula is equivalent to a Π_n-formula. Thus, $\mathrm{L}\Delta_n$ implies that $\theta(z, a, \boldsymbol{p})$ is Δ_n, and we can apply the least number principle to $\theta(z, a, \boldsymbol{p})$. Let z_0 be the least number z such that $\theta(z, a, \boldsymbol{p})$ and let u_0 be the least number u such that $(\exists \boldsymbol{y} < u)\varphi_0(z_0, \boldsymbol{y}, \boldsymbol{p})$.

We claim that $(\forall x < a)(\exists \boldsymbol{y} < u_0)\varphi_0(x, \boldsymbol{y}, \boldsymbol{p})$. By the definition of z_0, if x is greater than or equal to z_0, then there is a $\boldsymbol{y}$ such that $\boldsymbol{y} \leq u_0$ and $\varphi_0(x, \boldsymbol{y}, \boldsymbol{p})$. Suppose that there is an $x < a$ such that the least witness to $(\exists \boldsymbol{y})\varphi_0(x, \boldsymbol{y}, \boldsymbol{p})$ is greater than u_0. Then, consider the set of x's such that $(\forall \boldsymbol{y} < u_0)\neg\varphi_0(x, \boldsymbol{y}, \boldsymbol{p})$. Again by $\mathrm{B}\Sigma_{n-1}$, this set is Π_{n-1}. By $\mathrm{L}\Delta_n$, it has a greatest element z_1. (Think of the least number of the form $a - x$ for such an x). But z_1 would also satisfy $\theta(z, a, \boldsymbol{p})$ and be less than z_0, a contradiction. ■

The principle of induction for Δ_n-formulas ($\mathrm{I}\Delta_n$). $\mathrm{I}\Delta_n$ consists of all the sentences

$$(\forall \boldsymbol{p})\left[\begin{array}{c}(\forall x)(\varphi(x, \boldsymbol{p}) \leftrightarrow \neg\psi(x, \boldsymbol{p})) \rightarrow \\ \big((\varphi(0, \boldsymbol{p}) \wedge (\forall x)(\varphi(x, \boldsymbol{p}) \rightarrow \varphi(x+1, \boldsymbol{p}))) \rightarrow (\forall x)\varphi(x, \boldsymbol{p})\big)\end{array}\right]$$

in which φ and ψ are Σ_n.

Paris raised the question of determining the relationship between $\mathrm{I}\Delta_n$ and $\mathrm{B}\Sigma_n$; see Clote and Krajíček (1993).

As we will see in Section 2, the problem is to decide whether $\mathrm{B}\Sigma_n$ (or equivalently $\mathrm{L}\Delta_n$) follows from $\mathrm{I}\Delta_n$. There is a natural approach to showing that it does, but this approach does not lead to a correct proof. However, it does point out where the problem lies.

Let us attempt to prove $\mathrm{L}\Delta_n$ from $\mathrm{I}\Delta_n$. Suppose that we are given a nonempty Δ_n-set A, and let us attempt to show that it has a least element. We suppose that A has no least element, and we look for a failure of induction. Define the set

$$A^* = \{x : (\forall y)[y \leq x \rightarrow y \notin A]\}.$$

Clearly $0 \in A^*$, or 0 would be the least element of A. Equally clearly, if $a \in A^*$ and $a + 1 \notin A^*$, then $a + 1$ would be the least element of A^*. So, A^* satisfies the hypotheses needed to apply induction.

It would only remain to show that A^* is a Δ_n-set. Since A^* is explicitly Π_n, it would be sufficient to show that it is Σ_n. There is a standard argument to show that Σ_n-predicates are closed under bounded universal quantification. If φ is $(\exists w)\varphi_0$, where φ_0 is Π_{n-1}, then we would rewrite

$$(\forall y \leq x)(\exists w)\varphi_0$$

as

$$(\exists u)(\forall y \le x)(\exists w < u)\varphi_0,$$

and then replace $(\exists w < u)\varphi_0$ by a formula which is Π_{n-1}. But look at the implication,

$$(\forall y \le x)(\exists w)\varphi_0 \to (\exists u)(\forall y \le x)(\exists w < u)\varphi_0.$$

We have confronted an instance of $\mathrm{B}\Sigma_n$, rather than an instance of $\mathrm{B}\Sigma_{n-1}$ such as we found we found earlier. Of course, we cannot use an instance of $\mathrm{B}\Sigma_n$ to prove $\mathrm{B}\Sigma_n$, and the natural argument fails.

Even so, there is a less direct argument leading to the same conclusion. If we strengthen the base theory to include the assertion that exponentiation is a total function (exp), then we obtain the following partial answer to Paris's question.

$$\mathrm{PA}^- + \mathrm{I}\Sigma_0 + \exp \implies (\mathrm{I}\Delta_n \iff \mathrm{B}\Sigma_n)$$

2 Equating Bounding and Induction

2.1 $\mathrm{B}\Sigma_n$ implies $\mathrm{I}\Delta_n$.

For models of $\mathrm{PA}^- + \mathrm{I}\Sigma_0$, the implication

$$\mathfrak{M} \models \mathrm{B}\Sigma_n \implies \mathfrak{M} \models \mathrm{I}\Delta_n$$

is well known. However, the proof is short and so we include it here.

Suppose that $\mathfrak{M} \models \mathrm{B}\Sigma_n$, that φ and ψ are Σ_n-formulas, and that there are parameters $\boldsymbol{p}$ in $\mathfrak{M}$ relative to which φ and $\neg\psi$ define the same subset J of $\mathfrak{M}$. We argue that J is not a counterexample to $\mathrm{I}\Delta_n$ in $\mathfrak{M}$.

First, we can write φ as $\exists \boldsymbol{y}\varphi_0$ and ψ as $\exists \boldsymbol{y}\psi_0$, where φ_0 and ψ_0 are Π_{n-1}-formulas, which we may assume have the same number of free variables.

Now, suppose that a is strictly above some element of the complement of J in $\mathfrak{M}$. Since φ and ψ define complementary sets,

$$\mathfrak{M} \models (\forall x < a)(\exists \boldsymbol{y})[\varphi_0(x, \boldsymbol{y}, \boldsymbol{p}) \vee \psi_0(x, \boldsymbol{y}, \boldsymbol{p})].$$

By applying $\mathrm{B}\Sigma_n$ in $\mathfrak{M}$, there is a b in $\mathfrak{M}$ such that

$$\mathfrak{M} \models (\forall x < a)(\exists \boldsymbol{y} < b)[\varphi_0(x, \boldsymbol{y}, \boldsymbol{p}) \vee \psi_0(x, \boldsymbol{y}, \boldsymbol{p})].$$

But then, for the elements of $\mathfrak{M}$ which are less than a, J is defined by the formula $(\exists \boldsymbol{y} < b)\varphi_0(x, \boldsymbol{y}, \boldsymbol{p})$ in $\mathfrak{M}$. Using $\mathrm{B}\Sigma_n$ again, the intersection of J with the numbers below a is definable by a Π_{n-1}-formula relative to the parameters b, a, and $\boldsymbol{p}$. Since $\mathrm{B}\Sigma_n$ implies $\mathrm{I}\Pi_{n-1}$ and there is an element of the complement of J which is less than a, either $0 \notin J$ or J is not closed under the successor, as required.

2.2 $\mathrm{I}\Delta_n$ implies $\mathrm{B}\Sigma_n$

It is in our argument for the implication from $\mathrm{I}\Delta_n$ to $\mathrm{B}\Sigma_n$ that we make use of the exponential function. Ultimately, we take an element a in $\mathfrak{M}$, and use the base a representations of numbers less than a^a to code length a sequences of numbers less than a.

In the meantime, we make use of the existence of a standard coding within models $\mathfrak{M}$ of $\mathrm{PA}^- + \mathrm{I}\Delta_1 + \exp$ of sequences of elements of $\mathfrak{M}$ by elements of $\mathfrak{M}$ so that the relations "c codes a sequence of length i" and "c_j is the jth element of the sequence coded by c" are Δ_1 in $\mathfrak{M}$. We will write $\langle m_j : j < i\rangle$ to denote the code for the sequence of length i whose elements are the numbers m_j, for j less than i.

Theorem 2.1 *If $n \geq 1$ and $\mathfrak{M}$ is a model of $\mathrm{PA}^- + \mathrm{I}\Sigma_0 + \exp$, then*

$$\mathfrak{M} \models \mathrm{I}\Delta_n \Longrightarrow \mathfrak{M} \models \mathrm{B}\Sigma_n.$$

We will prove Theorem 2.1 for the case when $n = 1$. The general case for $n > 1$ follows by the same argument relative to the complete Σ_{n-1}-subset of $\mathfrak{M}$.

Lemma 2.2 *Suppose that $\mathfrak{M}$ is a model of $\mathrm{PA}^- + \mathrm{I}\Delta_1 + \exp$ and $\mathfrak{M}$ is not a model of $\mathrm{B}\Sigma_1$. There are an element $a \in \mathfrak{M}$ and a function $f : [0, a) \to \mathfrak{M}$ such that the following conditions hold.*

1. *f is injective.*
2. *The range of f is unbounded in $\mathfrak{M}$.*
3. *The graph of f is Σ_0 relative to parameters in $\mathfrak{M}$.*

Proof: By hypothesis, $\mathfrak{M}$ is a not a model of $\mathrm{B}\Sigma_1$. Consequently, we may fix $a \in \mathfrak{M}$, a sequence of parameters $\boldsymbol{p}$ from $\mathfrak{M}$, and a Σ_1-formula $(\exists \boldsymbol{w})\varphi_0$ such that φ_0 is a Σ_0-formula and the following conditions hold.

$$\mathfrak{M} \models (\forall x < a)(\exists \boldsymbol{y})(\exists \boldsymbol{w})\varphi_0(x, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{p}) \tag{1}$$
$$\mathfrak{M} \models (\forall s)(\exists x < a)(\forall \boldsymbol{y} < s)\neg(\exists \boldsymbol{w})\varphi_0(x, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{p}) \tag{2}$$

Define f so that for $x < a$, $f(x)$ is $\langle x, s_x\rangle$, where s_x is least s such that

$$\mathfrak{M} \models (\exists \boldsymbol{y} < s)(\exists \boldsymbol{w} < s)\varphi_0(x, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{p}). \tag{3}$$

There is an s satisfying Equation 3 by application of Equation 1, and there is a least such s by application of $\mathrm{I}\Delta_1$ in $\mathfrak{M}$. Clearly, f is an injective function, since x is determined by the first coordinate of $f(x)$. The range of f is unbounded by Equation 2. Finally, the graph of f is definable by a Σ_0-formula essentially because the value of f at x is chosen to bound all of the quantifiers used to define it from x. Notice, we are invoking the properties of a well behaved pairing function. ■

Lemma 2.3 *Suppose that $\mathfrak{M}$ is a model of $\mathrm{PA}^- + \mathrm{I}\Delta_1 + \exp$ and that $\mathfrak{M}$ is not a model of $\mathrm{B}\Sigma_1$. There are $a \in \mathfrak{M}$, a nonprincipal Σ_1-cut I, and a function g such that the following conditions hold.*

1. *$I \subset [0, a)$.*
2. *$g : I \to \mathfrak{M}$.*
3. *The graph of g is Σ_1 relative to parameters in $\mathfrak{M}$.*
4. *For each $i \in I$, $g(i)$ is the code for a sequence $\langle m_j : j < i \rangle$ such that for all unequal j_1 and j_2 less than i, $m_{j_1} \neq m_{j_2}$. (Note that the sequence coded by $g(i)$ has length i.)*
5. *For each $i_1 < i_2 \in I$ the sequence coded by $g(i_2)$ is an end extension of the one coded by $g(i_1)$. In other words, if $g(i_1)$ codes $\langle m_j : j < i_1 \rangle$ and $g(i_2)$ codes $\langle n_j : j < i_2 \rangle$, then for all $j < i_1$, $m_j = n_j$.*
6. *For each $m < a$, there is an $i \in I$ such that m appears in the sequence coded by $g(i)$.*

Proof: Let f and a be as in Lemma 2.2. We view the set of numbers less than a, which is finite in the sense of $\mathfrak{M}$, as a recursively enumerable set in the sense of $\mathfrak{M}$. In this sense, m is enumerated into this set at stage $f(m)$. Since f is injective, there is at most one number enumerated during each stage. At each stage s, we can define the sequence of numbers which have been enumerated at earlier stages and order them according to the order in which they were enumerated by f. That is, we reorder the numbers n and m below a such that n comes before m if and only if $f(n)$ is less than $f(m)$.

We define g so that $g(i)$ is the code for sequence $\langle m_j : j < i \rangle$ such that for each $j < i$, m_j is the jth number enumerated by f. More formally, $g(i) = \langle m_j : j < i \rangle$ if and only if there is an s such that the following conditions hold.

i. $\{m : (\exists y < s)(f(m) = y)\} = \{m : (\exists j < i)(m = m_j)\}$

ii. For all j_1 and j_2 less than i, $(j_1 < j_2) \leftrightarrow (f(m_{j_1}) < f(m_{j_2}))$.

We leave it to the reader to verify that g is well defined and Σ_1 in $\mathfrak{M}$, relative to the parameters needed to define f.

Let I denote the domain of g.

By definition, for each i in I, there is an s in $\mathfrak{M}$ so that there are at least i many elements m of $\mathfrak{M}$ such that $f(m) < s$. For such an s and any j less than i, there is a smaller s_j which is adequate to define g at j, so I is an initial segment of $\mathfrak{M}$. In fact, the map sending i to m_{i-1} is an order isomorphism between $I \setminus \{0\}$ and the range of f. Further, for each $i \in I \setminus \{0\}$, the restriction of this map to $\{j : 0 < j \leq i\}$ is coded within $\mathfrak{M}$. Consequently, ordering $I \setminus \{0\}$ and the range of f by the ordering of $\mathfrak{M}$ produces isomorphic order types.

Now, suppose that $i > a$ in $\mathfrak{M}$. If i were to be in I, then $g(i)$ would be the code for a sequence of length greater than a, with elements less than a, and with no repetitions. This is impossible, since every model of $\mathrm{PA}^- + \mathrm{I}\Delta_1 + \exp$ satisfies the Σ_0-pigeon hole principle; see Hájek and Pudlák (1998). Thus, I is a subset of $[0, a)$. Finally, since the range of f is unbounded in $\mathfrak{M}$ and I has the same order type as the range of f, there can be no greatest element of I. Consequently, I is a proper nonprincipal cut in $\mathfrak{M}$. I is Σ_1 by virtue of being the domain of the Σ_1 function g.

Finally, for every m in the domain of f and every sufficiently large i in I, m appears in the sequence coded by $g(i)$. Since the domain of f is $[0, a)$, for each $m < a$, there is an $i \in I$ such that m appears in the sequence coded by $g(i)$. ■

Let g and I be fixed as in Lemma 2.3. Let $\boldsymbol{m}^* = \langle m_i^* : i \in I \rangle$ be the sequence of length I such that for all $i \in I$, m_i^* is equal to the ith element of $g(i+1)$. That is, $\boldsymbol{m}^*$ is the sequence given by the limit of the range of g.

Lemma 2.4 *Suppose that $c \in \mathfrak{M}$, $\boldsymbol{n} = \langle n_j : j < c \rangle$ is coded in $\mathfrak{M}$, and $\boldsymbol{n}$ is a sequence of elements of $\mathfrak{M}$ which are less than a. Then, either c is not an upper bound for I or there is an $i \in I$ such that $n_i \neq m_i^*$.*

Proof: Let c and $\boldsymbol{n} = \langle n_i : i < c \rangle$ be given as above, and suppose that c is an upper bound for I. For the sake of a contradiction, suppose that for all $i \in I$, $m_i^* = n_i$.

But then consider the set

$$J = \{j : \text{For all } i < j, n_i \neq n_j.\}$$

For each i in I, the sequence coded by $g(i)$ has no repeated values, and so $I \subseteq J$. Conversely, every element of $\mathfrak{M}$ which is less than a appears in $\boldsymbol{m}^*$. Consequently, if $j < c$ and $j \notin I$, then there is an i in I such that $m_i^* = n_j$. But then, $n_i = n_j$ and so j is not an element of J. Thus, J is equal to I.

But then J is a proper Σ_0-cut, contrary to the assumption that $\mathfrak{M}$ is a model of Δ_1-induction. ■

We can now present the proof of Theorem 2.1.

Proof: Suppose that $\mathfrak{M}$ is a model of $\mathrm{PA}^- + \mathrm{I}\Delta_1 + \exp$ and that $\mathfrak{M}$ is not a model of $\mathrm{B}\Sigma_1$. Let I, g, and $\boldsymbol{m}^*$ be fixed as above.

We define a collection of intervals $[c_i, d_i]$, for $i \in I$. If $i = 0$, then $c_0 = 0$ and $d_0 = a^a$. For $i > 0$,

$$c_i = \sum_{j<i} m_j^* a^{a-(j+1)} \text{ and } d_i = c_i + a^{a-i}.$$

We calculate the first few values explicitly.

$$[c_0, d_0] = [0, a^a]$$
$$[c_1, d_1] = [m_0^* a^{a-1}, (m_0^* + 1)a^{a-1}]$$
$$[c_2, d_2] = [m_0^* a^{a-1} + m_1^* a^{a-2}, m_0^* a^{a-1} + (m_1^* + 1)a^{a-2}]$$

Since each m_j^* is less than a, for each $i \in I$, $[c_{i+1}, d_{i+1}] \subseteq [c_i, d_i]$. Define J by

$$x \in J \iff (\exists i)(x \leq c_i),$$

and define K by

$$x \in K \iff (\exists i)(x \geq d_i).$$

Since the initial segments of $\boldsymbol{m}^*$ are Σ_1-definable relative to parameters in $\mathfrak{M}$, J and K are Σ_1-definable relative to parameters in $\mathfrak{M}$. In addition, J is closed downward in $\mathfrak{M}$, and K is closed upward. It remains to show that J is a proper cut and that $J = \mathfrak{M} \setminus K$.

Since there is only one $i \in I$ such that m^*_i is equal to 0, for all but one i, $c_i < c_{i+1}$. Since I has no greatest element, it follows that neither does J. Consequently, J is a proper cut.

Now, suppose that n is an element of $\mathfrak{M}$ which is neither an element of J nor one of K. Then, consider the base-a representation of n. Let n_i be the coefficient of the a^{a-i-1} term in this representation. For example, if n where $2a^{a-1} + 3a^{a-2}$, then n_0 would be 2, n_1 would be 3, and for every other i less than a, n_i would be 0. By the choice of n, for every i in I,

$$\sum_{j<i} m^*_j a^{a-(j+1)} < n < \sum_{j<i} m^*_j a^{a-(j+1)} + a^{a-i}$$

But then, for each i in I, n_i is equal to m^*_i. Since the base-a representation of n is definable from n by a bounded recursion, $\mathrm{I}\Delta_1$ implies that the sequence $\langle n_j : j < a\rangle$ is coded by an element of $\mathfrak{M}$. This is a contradiction to Lemma 2.4.

Consequently, J is a Σ_1-cut and the Σ_1-set K is its complement in $\mathfrak{M}$. So, from the failure of $\mathrm{B}\Sigma_1$ in $\mathfrak{M}$, we produced a failure of $\mathrm{I}\Delta_1$ in $\mathfrak{M}$. Theorem 2.1 follows. ■

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