Name and student ID number: ______________________________

GSI name and discussion section time: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	10	10	10	10	10	10	60
Score:							

We will grade all 6 of your solutions and drop your lowest score so that your total exam score will be out of 50.

1. [10 points] Decide if each assertion is always True or sometimes False. You do not need to provide any justification for your answer.

   (a) ___ If $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are linearly dependent, then $\text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

   (b) ___ If $S \subset \mathbb{R}^n$ is a spanning set, then any set $S' \subset \mathbb{R}^n$ containing S is a spanning set.

   (c) ___ If non-zero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal to each other, then $\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

   (d) ___ Given vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^2$, there is a unique 2×2 matrix A such that $A\mathbf{v}_1 = \mathbf{w}_1, A\mathbf{v}_2 = \mathbf{w}_2$.

   (e) ___ If an $m \times n$ matrix A has null space $\text{Null}A = \{\mathbf{0}\}$, then A has column space $\text{Col}A = \mathbb{R}^m$.

2. [10 points] Consider the affine subspace $W = \{z = 1\} \subset \mathbb{R}^3$. Your friend impishly parametrizes it using s, t and writing vectors $\mathbf{w} \in W$ in the form

$$\mathbf{w} = \mathbf{v}_0 + s\mathbf{v}_1 + t\mathbf{v}_2, \qquad \text{where} \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Answer each of the following questions about this situation.

(a) What vector $\mathbf{w}$ corresponds to $s = 2, t = -3$?

(b) What are s and t for the vector

$$\mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}?$$

(c) Find an equation in s and t describing the affine subspace $\{x = y, z = 1\} \subset W$.

(d) What are s and t for the point where W intersects the line

$$\text{Span}\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\}?$$

(e) In the triangle with vertices given by the three points

$$s = 1, t = 0 \quad s = 0, t = 1 \quad s = 1, t = 1$$

what is the interior angle at the point $s = 1, t = 1$?

3. [10 points] For each of the following, find a matrix A satisfying the given conditions or explain why it is impossible for such an A to exist.

(a) $\text{Null}A = \{x + y = 0\} \subset \mathbb{R}^2$, $\text{Im}A = \text{Span}\{\mathbf{e}_1 + \mathbf{e}_2\} \subset \mathbb{R}^2$.

(b) $\text{Null}A = \{-x + y - z = 0\} \subset \mathbb{R}^3$, $\text{Im}A = \text{Span}\{\mathbf{e}_1 - \mathbf{e}_2\} \subset \mathbb{R}^2$.

(c) $\text{Null}A = \{x - y = 0\} \subset \mathbb{R}^2$, $\text{Im}A = \text{Span}\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3\} \subset \mathbb{R}^3$.

(d) $\text{Null}A = \{x + y = 0, y + z = 0\} \subset \mathbb{R}^3$, $\text{Im}A = \text{Span}\{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\} \subset \mathbb{R}^3$.

(e) $\text{Null}A = \{x + y = 0, y + z = 0\} \subset \mathbb{R}^3$, $\text{Im}A = \text{Span}\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3\} \subset \mathbb{R}^3$.

4. [10 points] In $\mathbb{R}^2$, we have some data points that are apples

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

and some data points that are oranges

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find a linear function $f : \mathbb{R}^2 \to \mathbb{R}$ that is strictly positive on apples and strictly negative on oranges.

5. [10 points] For the following vectors, find all i such that $\mathbf{v}_i \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}\}$.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 4 \end{bmatrix}$$

6. [10 points] For $t \in \mathbb{R}$, consider the matrix

$$A = \begin{bmatrix} t & 1 & t^2 \\ 1 & 1 & 0 \\ t^2 & 1 & t \end{bmatrix}$$

For each $t \in \mathbb{R}$, say whether the image $\operatorname{Im} A$ is a point, line, plane or all of space, and find a basis of $\operatorname{Im} A$.