MATH 54: MIDTERM 2, V.A: SOLUTIONS

Question 1 (10 points)**.** Mark each statement as "True" or "False". Give appropriate justification for your answer in 1 to 2 sentences.

(i) For $n \geqslant m$, let A be an $m \times n$ matrix with a pivot in every row. Then $\dim(\text{nul}(A)) = n - m$.
(ii) If $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{C} = \{c_1, c_2, c_3\}$ are bases for $\mathbb{R}^3$, then so is $\{b_1 + c_1, b_2 + c_2, b_3 + c_3\}$.
(iii) The dimension of $\text{nul}(A)$ is the algebraic multiplicity of 0 as an eigenvalue of A.
(iv) The angle between

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

is $\pi/3$.
(v) If an $n \times n$ matrix A can be factored as $A = QR$ where Q is orthogonal, then $\det(A) = \pm\det(R)$.

Solution (i)

True. The rank of A is m, so by rank-nullity, $n = m + \dim(\text{nul}(A))$. Rearranging this equality yields the claim.

Solution (ii)

False. Take $\mathcal{B} = \{e_1, e_2, e_3\}$ and $\mathcal{C} = \{-e_1, -e_2, -e_3\}$. If we sum the bases element-wise, we just have $\{0\}$, which is not a basis of $\mathbb{R}^3$.

Solution (iii)

False.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is a counterexample, as $\dim(\text{nul}(A)) = 1$, but 0 has multiplicity 2 as a eigenvalue of A.

Solution (iv)

False. We can just compute the angle directly:

$$\arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$$

Solution (v)

True. We know that $\det(A) = \det(Q)\det(R)$ and that the determinant of an orthogonal matrix ($\det(Q)$) must be either 1 or -1.

Question 2 (10 points). Find a basis of the vector space $\{Av : v \in \mathbb{R}^4\}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Solution

Note that the vector space in questions is the column space of A. We begin by row reducing A.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we can see that the first two columns span the column space of A, so a basis is

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$$

Question 3 (10 points)**.** Consider the vector space $V = \{p(t) \in \mathbb{P}_3 : p(-1) = 0\}$.

(i) Show that $\mathcal{B} = \{1 + t, t + t^2, t^2 + t^3\}$ is a basis for V.

(ii) Compute $[4 + 6t + 8t^2 + 6t^3]_{\mathcal{B}}$.

Solution (i)

To see that $\mathcal{B}$ is linearly independent, suppose that $c_1(1+t)+c_2(t+t^2)+c_3(t^2+t^3) = 0$. Rearranging the LHS, we have $c_1 + (c_1 + c_2)t + (c_2 + c_3)t^2 + c_3t^3 = 0$. This implies that each c_i is 0. On the other hand, V is a subspace of $\mathbb{P}_3$, which is 4-dimensional. Since $V \neq \mathbb{P}_3$ (for instance, $1 \in P_3$, but $1 \notin V$), we know that $\dim(V) \leqslant 3$. Hence, $\mathcal{B}$ spans V, so $\mathcal{B}$ is a basis for V.

Solution (ii)

We write $4 + 6t + 8t^2 + 8t^2 + 6t^3 = c_1(1 + t) + c_2(t + t^2) + c_3(t^2 + t^3)$. We can immediately extract that $c_1 = 4$ and $c_3 = 6$. Looking at the coefficient for the linear term, c_2 is then 2. Therefore,

$$[4 + 6t + 8t^2 + 6t^3]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$

Question 4 (10 points)**.** For each part, determine whether A and B are similar. If they are, find a matrix P such that $A = PBP^{-1}$. If not, explain why in a couple sentences.

(i)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

(ii)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

Solution (i)

These matrices are similar since they have the same eigenvalues and are diagonalizable. As B is a diagonal matrix, the matrix P that we are after should be composed of the eigenvalues of A. The eigenvector of A corresponding to 1 is just e_1. The eigenvector corresponding to 2 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, as both rows sum to 2. Therefore, we can take $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution (ii)

These matrices are not similar, since they have different eigenvalues: 2 is an eigenvalue of A (both rows sum to 2), but it is not an eigenvalue of B, since

$$B - 2I = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

is invertible.

Question 5 (10 points). Let $V = \operatorname{span}(\{v_1, v_2, v_3\})$ where

$$v_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad v_3 = \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \quad \text{and} \quad v = \begin{bmatrix}2\\2\\1\\-1\end{bmatrix}$$

(i) Find an orthogonal basis for V by perfoming Gram-Schmidt on the ordered basis $\{v_1, v_2, v_3\}$.
(ii) Find the orthogonal projection of v onto V.

Solution (i)

We simply execute the Gram–Schmidt algorithm.

$$w_1 : \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

$$w_2 : \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{10}{30}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}2/3\\1/3\\0\\-1/3\end{bmatrix} \rightsquigarrow \begin{bmatrix}2\\1\\0\\-1\end{bmatrix}$$

$$w_3 : \begin{bmatrix}0\\1\\0\\1\end{bmatrix} - \frac{6}{30}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} - \frac{0}{6}\begin{bmatrix}2\\1\\0\\-1\end{bmatrix} = \begin{bmatrix}-1/5\\3/5\\-3/5\\1/5\end{bmatrix} \rightsquigarrow \begin{bmatrix}-1\\3\\-3\\1\end{bmatrix}$$

Therefore, one basis orthogonal basis for V is

$$\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}2\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\3\\-3\\1\end{bmatrix}\right\}$$

Solution (ii)

The projection can be computed as follows:

$$\operatorname{proj}_V(v) = \frac{3}{30}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \frac{7}{6}\begin{bmatrix}2\\1\\0\\-1\end{bmatrix} + \frac{0}{20}\begin{bmatrix}-1\\3\\-3\\1\end{bmatrix} = \begin{bmatrix}73/30\\41/30\\3/10\\-23/30\end{bmatrix}$$

Question 6 (10 points)**.** Consider the matrix

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Use diagonalization over $\mathbb{C}$ to compute A^{1002}.

Solution

Theorem 9 from §5.5 tells us that the eigenvalues of A are $1/\sqrt{2} \pm i/\sqrt{2}$. First we find the corresponding eigenvectors. If $\begin{bmatrix} a \\ b \end{bmatrix}$ is an eigenvector for $1/\sqrt{2} + i/\sqrt{2}$, we obtain the equation

$$a/\sqrt{2} - b/\sqrt{2} = a/\sqrt{2} + ai/\sqrt{2}$$

so $ai = -b$. We take the eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$. An eigenvector for the $1/\sqrt{2} - i/\sqrt{2}$ will then be the complex conjugate of the first eigenvector we found: $\begin{bmatrix} 1 \\ i \end{bmatrix}$. This means that

$$A = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} + i/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} - i/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1}$$

We can rewrite the entries in the diagonal polar form to make it easier to take a high power.

$$D = \begin{bmatrix} 1/\sqrt{2} + i/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} - i/\sqrt{2} \end{bmatrix} = \begin{bmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} \rightsquigarrow D^{1002} = \begin{bmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Where we use the facts that $(e^{i\pi/4})^8 = (e^{-i\pi/4})^8 = 1$ and that $1002 = 2 + 125 \cdot 8$. Now we compute the inverse of the change-of-basis matrix that we found.

$$\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} = \frac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{bmatrix}$$

Finally, we are ready to compute A^{1002}!

$$\begin{aligned} A^{1002} &= \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{bmatrix} \\ &= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2i \\ 1/2 & 1/2i \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

Question 7 (10 points)**.** Let $M_2(\mathbb{R}) = \{\text{the vector space of all } 2 \times 2 \text{ real matrices}\}$. For some $B \in M_2(\mathbb{R})$, suppose that $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is given by $T(A) = BA + A^\top$. If $\dim(\ker(T)) = 3$, find B.

(HINT: Put $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and fix a basis $\mathcal{B}$ for $M_2(\mathbb{R})$. Then compute $[T]_\mathcal{B}$ in terms of a, b, c, and d.)

Solution

Let

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

We want to find $[T]_\mathcal{B}$, so we apply T to each of the basis vectors:

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+1 & 0 \\ c & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 1 & c \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b & 1 \\ d & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d+1 \end{bmatrix}$$

Therefore,

$$[T]_\mathcal{B} = \begin{bmatrix} a+1 & 0 & b & 0 \\ 0 & a & 1 & b \\ c & 1 & d & 0 \\ 0 & c & 0 & d+1 \end{bmatrix}$$

We want this matrix to be a rank 1 matrix, which will only obtain when $a = d = -1$ and $b = c = 0$. Therefore

$$B = -I_2$$