MATH 54 MIDTERM 1 SOLUTIONS

Question 1. Mark each statement as "True" or "False." If "True," briefly explain why (1-3 sentences). If "False," briefly explain why (1-3 sentences) OR give a counterexample.

(i) Suppose A is an $m \times n$ matrix and B is an $n \times m$ matrix such that

$$BA = I_n,$$

where I_n is the $n \times n$ identity matrix. Then every linear system $A\mathbf{x} = \mathbf{b}$ has at most one solution.

(ii) The span of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the same set as the span of the vectors $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$.

(iii) For invertible $n \times n$ matrices A, B,

$$\left(A^T B\right)^{-1} = \left((B^{-1})^T A^{-1}\right)^T$$

(iv) The matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

defines a linear transformation which in particular takes vectors in the yz-plane (or x_2x_3-plane) to other vectors in the yz-plane (or x_2x_3-plane).

(v) The span of the columns of A is the same set as the span of the columns of $REF(A)$.

(vi) If A is a square matrix and $\mathbf{b}$ is a vector such that $A^2\mathbf{x} = \mathbf{b}$ has a solution, then $A\mathbf{x} = \mathbf{b}$ has a solution.

Solution.

(i) $\boxed{\text{True}}$. The condition that every linear system $A\mathbf{x} = \mathbf{b}$ has at most one solution is equivalent to the statement that A defines an injective linear transformation. This in turn is the same as A having linearly independent columns, which is the same as $A\mathbf{x} = \mathbf{0}$ having only $\mathbf{0}$ as a solution. This is assured by the assumption that $BA = I_n$, since if $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = BA\mathbf{x} = \mathbf{0}$.

(ii) $\boxed{\text{True}}$. To check that the spans of two collections of vectors are equal, it suffices to check that each vector in each collection lies in the span of the other collection. Call S_1, S_2 the two spans above. Since

$$\mathbf{v}_1 = \frac{1}{2}((\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{v}_1 - \mathbf{v}_2)),$$
$$\mathbf{v}_2 = \frac{1}{2}((\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{v}_1 - \mathbf{v}_2)),$$

it means that $\mathbf{v}_1, \mathbf{v}_2$ lie in S_2. The other direction is similar.

(iii) $\boxed{\text{False}}$. Simplify and compare both sides:

$$(A^T B)^{-1} = B^{-1}(A^T)^{-1},$$
$$((B^{-1})^T A^{-1})^T = (A^{-1})^T((B^{-1})^T)^T = (A^T)^{-1}B^{-1}$$

The A and B are in flipped orders, so the two expressions aren't equal in general since matrix multiplication is not commutative.

(iv) $\boxed{\text{False}}$. The give matrix swaps the x- and z-axes, and scales the y-axis by a factor of 4. So in particular, it would take the vector $\mathbf{e}_3$ inside the yz-plane to the vector $\mathbf{e}_1$ outside the yz-plane.

(v) $\boxed{\text{False}}$. While it is true that row-reduction of a matrix preserves the nullspace, it does not preserve the span of the columns. Take for example $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. $REF(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so the column span of A is the y-axis, and the column span of $REF(A)$ is the x-axis.

(vi) $\boxed{\text{True}}$. If $AA\mathbf{x} = \mathbf{b}$ has a solution, called $\mathbf{x}_0$, then $A(A\mathbf{x}_0) = \mathbf{b}$, so the vector $A\mathbf{x}_0$ solves $A\mathbf{y} = \mathbf{b}$.

Question 2. Let A be the matrix

$$\begin{bmatrix} 1 & 5 & 2 \\ 2 & 11 & -2 \\ 3 & 13 & 18 \end{bmatrix}.$$

(i) For which real values of k does the equation $A\mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ k \end{bmatrix}$ have solutions?

(ii) Find all solutions to $A\mathbf{x} = \mathbf{0}$.

(iii) Find *any* solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(iv) Find the general solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. [Hint: using your answers to (ii) and (iii), you don't have to do any more work if you use a theorem.]

Solution.

(i) Let's row reduce the system

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 2 & 11 & -2 & -1 \\ 3 & 13 & 18 & k \end{array}\right]$$

Let's subtract 2 times the first row from the second. We get:

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 1 & -6 & -1 \\ 3 & 13 & 18 & k \end{array}\right]$$

Next subtract 3 times the first row from the third:

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 1 & -6 & -1 \\ 0 & -2 & 12 & k \end{array}\right]$$

Lastly, let's add 2 times the second row to the third row:

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 1 & -6 & -1 \\ 0 & 0 & 0 & k-2 \end{array}\right]$$

And we see that in order for this system to have solutions, we $\boxed{\text{need } k = 2}$.

(ii) Let's row reduce again; we can actually just take our work above since the row reductions will be the same. Since the augmentation column is all zeros now, none of our row reductions will change this column! At the end of our row reduction, we get, just like above,

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

And we see we have a free variable x_3; and we need $x_2 = 6x_3$ and $x_1 = -5x_2 - 2x_3 = -32x_3$. In vector form, our solution is

$$\begin{bmatrix} -32x_3 \\ 6x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -32 \\ 6 \\ 1 \end{bmatrix},$$

where x_3 is any real number. We can check our work by multiplying our solution with A, and we do get 0.

(iii) The problem just says to find *any* solution. Namely, notice that the first column of A is already this $\mathbf{b}$ vector! So we can $\boxed{\text{take } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$ as our solution, as multiplying by this vector just "catches" the first column. It weighs the first column by 1, and weighs the second and third by 0, so they drop out.

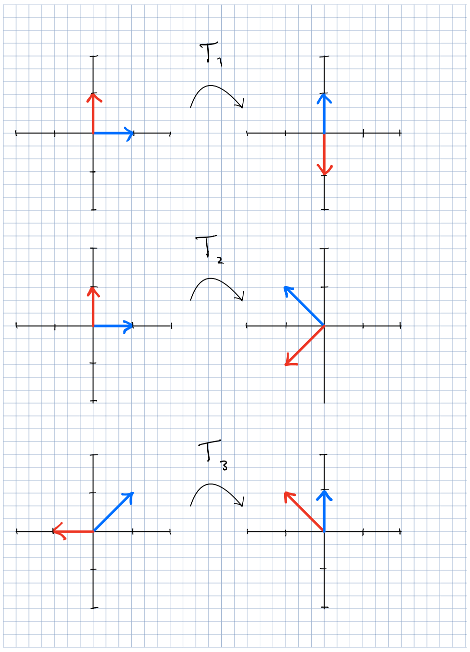
(iv) By the solution principle; ie the principle that if $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$, they every solution to the given equation is uniquely of the form $\mathbf{p} + \mathbf{h}$ where $\mathbf{h}$ is a solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. So, our general solution can be gotten just from taking our particular solution from part (iii) and the homogeneous solution from part (ii) and adding them together: $\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -32 \\ 6 \\ 1 \end{bmatrix}, \text{ where } c \text{ is any scalar}}$. This is our general solution! You can check by multiplying it by A, we should get $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a result.

You can also do this by row reduction, in which case the row reduction should look very similar to our solution in part (i). The answer you get might look a little different from the solution you get here, but remember, this piece sticking out, the $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for our solution here is not unique. You could add any $c \begin{bmatrix} -32 \\ 6 \\ 1 \end{bmatrix}$ to it for some fixed number c and get an equivalent general solution. This is since we are describing a line, and you can start from *any* point on the line and then add the "slope" term and get the same line.

Question 3. Answer the following two parts.

(i) Below is an image of the effects of three linear transformations T_1, T_2, T_3 on certain vectors in $\mathbb{R}^2$. Here, the tick marks on the axes are spaced 1 unit apart, and each T_i sends the blue arrow to the blue arrow, and the red arrow to the red arrow.

Write down the standard matrix of each linear transformation, and briefly justify your response.



(ii) The following presents some possibly partial information about three maps $R, S, T : \mathbb{R}^3 \to \mathbb{R}^3$. First, decide whether this information (1) leaves open the possibility that the given map is linear, or (2) makes the map not linear. Second, justify your decision: if (1), give an example of such a linear transformation, and if (2), explain why such a map cannot be linear.

$$R\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} b+1+c \\ c-2a \\ a+c \end{bmatrix},$$

$$S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right),$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solution.

(i) In general, the standard matrix of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is determined by applying T to the standard basis vectors. If A is the standard matrix, then the ith column of A is $T(\mathbf{e}_i)$.

The map T_1 sends $\mathbf{e}_1 \mapsto \mathbf{e}_2$ and $\mathbf{e}_2 \mapsto -\mathbf{e}_2$, so the corresponding matrix is

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

The map T_2 sends $\mathbf{e}_1 \mapsto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{e}_2 \mapsto \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, so the corresponding matrix is

$$A_2 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

The map T_3 sends $-\mathbf{e}_1 \mapsto -\mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_1 + \mathbf{e}_2 \mapsto \mathbf{e}_2$. So, we know that $\mathbf{e}_1 \mapsto \mathbf{e}_1 - \mathbf{e}_2$, but we still need to figure out what it does to $\mathbf{e}_2$. To determine this, we can set up a system of equations, but here it's just easier to use linearity and note that $\mathbf{e}_2 = (\mathbf{e}_1 + \mathbf{e}_2) + (-\mathbf{e}_1)$, and we know where each one of those go. So $\mathbf{e}_2 \mapsto -\mathbf{e}_1 + 2\mathbf{e}_2$, and therefore the corresponding matrix is

$$A_3 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

(ii) $\boxed{R \text{ cannot be linear}}$, since there is a "+1" in the first row of the output, and therefore e.g. $R(\mathbf{0}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \mathbf{0}$, but a linear map must always send $\mathbf{0} \mapsto \mathbf{0}$.

$\boxed{S \text{ can be linear}}$. For example, it could be represented by the matrix $S = \mathbf{0}$.

$\boxed{T \text{ cannot be linear}}$, since $T(2\mathbf{e}_1) \neq 2T(\mathbf{e}_1)$

Question 4. Answer the following three parts.

(i) Suppose that the vectors $\begin{bmatrix} 1 \\ c \end{bmatrix}$ and $\begin{bmatrix} 5 \\ d \end{bmatrix}$ span all of $\mathbb{R}^2$. What can you say about the coefficients c and d?

(ii) For each n, give an example of n vectors in $\mathbb{R}^n$ that span $\mathbb{R}^n$, and verify this. In addition, explain why the vectors in your example are linearly independent.

(iii) Give an argument to show, if n vectors in $\mathbb{R}^n$ *do not* span $\mathbb{R}^n$, then they must be linearly dependent.

Solution.

(i) These vectors span $\mathbb{R}^2$ iff the REF of the matrix $\begin{bmatrix} 1 & 5 \\ c & d \end{bmatrix}$ has a pivot in every row. Since the matrix is square, this is equivalent to its REF having a pivot in every column, which is the condition that the columns are linearly independent. Two nonzero vectors are linearly independent iff they are not scalar multiples of one another. In this problem, this is true exactly when $\boxed{5c \neq d}$.

(ii) The n standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ span $\mathbb{R}^n$, since any vector $\mathbf{x} \in \mathbb{R}^n$ can be written as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$$

They are linearly independent because the only way such a sum can be the zero vector is if all $x_1 = x_2 = \cdots = x_n = 0$.

(iii) Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n vectors in $\mathbb{R}^n$ that do NOT span $\mathbb{R}^n$. Arrange them into the columns of the $n \times n$ matrix A. Since they don't span $\mathbb{R}^n$, it means that $REF(A)$ has fewer than n pivots. But then, it has fewer than n pivot columns, meaning it must have a free variable column. Thus there are nontrivial solutions $A\mathbf{x} = \mathbf{0}$, which means that the columns are linearly dependent.

Question 5. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

(i) Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly independent?

(ii) Let $B = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ be the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2$. Given any other vector $\mathbf{b} \in \mathbb{R}^3$, how many solutions can $B\mathbf{x} = \mathbf{b}$ have?

(iii) Take a matrix M whose columns are linearly independent, and let T be the linear transformation it defines. Must T be injective? Surjective? In both cases, justify your answer or give a counterexample.

Solution.

(i) Let's make the matrix with columns v_1, v_2, v_3 and row reduce! We want to find whether there is a pivot in each column, which will tell us whether the vectors are linearly independent or not.

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

We can add the first row to the second and get

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & -2 & 0 \end{bmatrix}$$

Then we can add the second row to the third:

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

And we see there's a pivot in every column (you can divide row 2 by 2 and row 3 by 4). Thus the vectors are linearly independent! You could also have checked this directly by definition; try solving the system

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}$$

which just comes down to doing the row reduction we did above.

(ii) The columns of B are still linearly independent! This follows either from the above row reduction in (i) where you can just ignore the third column (and this is a row reduction for B) or you can state the result that given a set of linearly independent vectors, any subset is also linearly independent. Anyway, since B's columns are linearly independent, it must have a pivot in each column, so it must have two pivots. It can't have a pivot in each row since it has three rows! So $B\mathbf{x} = \mathbf{b}$ has either 0 or 1 solution.

(iii) Given a matrix M whose columns are linearly independent, it doesn't have to be surjective. Just take the matrix B above! It doesn't have a pivot in each row although it does have a pivot in each column. M *does* have to be injective because it has a pivot in each column (it's columns are linearly independent!), so $M\mathbf{x} = \mathbf{b}$ has either 1 or 0 solutions.

Question 6. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Is A invertible? If so, find the inverse. If not, explain why not.
(ii) If M is invertible, is it possible that $M^k = 0$ for some positive integer k? Justify your answer.
(iii) Let M, N be square matrices, and suppose that M has inverse M^{-1} and MN has inverse D. Show that N is invertible, and give a formula for N^{-1} in terms of D, N, M, and M^{-1}. [Note that you cannot use N^{-1} in your proof since you don't yet know that it exists!]

Solution.

(i) A is in fact invertible. To find the inverse, let's use the row reduction algorithm:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Let's subtract 3 times the third row from the second, then 6 times the third from the first.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -6 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Finally let's subtract 3 times the second row from the first:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 3 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Thus the inverse is $\begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$. You can check your work by multiplying this by A, or even just picking one select row from A and one select column from A^{-1}; for example if we pick the second row from A and mulitply the first row in A^{-1} we should get 0, which is the $(2, 1)$ entry in the identity matrix I. And we do!

(ii) If M is invertible, M^k cannot be 0. This is because M invertible implies that M^k is also invertible! A formula for the inverse is $(M^k)^{-1} = (M^{-1})^k$. So M^k, being invertible, cannot be 0, since the 0 matrix is not invertible.

(iii) *Intuition:* This is my thought process, how I got the answer below. Let's cheat a bit, and assume N^{-1} did exist. Then $D = N^{-1}M^{-1}$. We want to solve for N^{-1}, so clearly we see $N^{-1} = DM$. However this derivation uses N^{-1}, which we aren't allowed to do! But now, intuitively we know that N^{-1} should be DM. Let's prove that it is.

To do so, we need to show that $(DM)N = I$ and $N(DM) = I$. In the first case, note $(DM)N = D(MN) = I$, by associativity and because D is an inverse to MN. In the second, note $MND = I$ holds because D is an inverse to MN. Now notice that:

$$\begin{aligned} MND &= I \\ ND &= M^{-1} \\ NDM &= I \end{aligned}$$

To move from the first equation to the second, we multiplied both sides on the left by M^{-1} Second to third, multiply both sides by M on the right. So we get our original equation $N(DM) = I$ holds!

Thus we showed that $\boxed{DM \text{ is the inverse to } N}$! So N is invertible.

Actually, if you use a theorem in the book, theorem 8 of section 2.3, you *just* need to check one of the two equations! So checking $(DM)N = I$ is enough, which was the easier one.