MATH 54 FINAL EXAM

Instructions:

- You have three hours to complete and upload this exam.
- At the beginning of your exam, be sure to write out the honor code and **sign it with your name**. *Unsigned exams will receive no credit.*
- You are allowed to consult: any notes from lecture or discussion taken by you or any of your instructors, any bCourses materials for the class like quiz solutions, the textbook, and Piazza. Do not use any calculators.
- Clearly mark your final answers, and show your work. If your work is missing or is difficult to follow, you may not receive credit even for a correct final answer!

Scanning and Upload Instructions:

- There are seven questions on this exam. Write your answer to each question on two pages (use one page if possible). If possible, please print out this template and upload your answers on this template. Otherwise, please upload your questions in order with exactly two pages per question (including any blank pages necessary), and a separate title page at the beginning. If you need to use the second page for any question, please note on the first page of the question that you have done so.
- If you *must* include extra pages, please include them at the end of your scan and clearly note in the corresponding solution that you have done so. Try to avoid using extra pages if possible.
- Please check each question in your scan after uploading to make sure that it has uploaded correctly.

In the box below (or on the title page of your submission if not writing directly on this template), **write out the following sentence** and **sign it with your name**:
"I have read and understood the instructions. By submitting my solutions, I agree that I have followed both the instructions and the Berkeley Honor Code."

Question 1. (True or False, **no reasonings required**)

(i) If a 2×2 matrix has eigenvalues 2 and 3, then its characteristic polynomial is $\lambda^2 + 2\lambda + 3$.

(ii) Let V be the vector space of 2×2 real symmetric matrices. There exists an injective linear transformation $T : \mathbb{P}_3 \to V$.

(iii) If f is a continuous odd function defined on $[-\pi, \pi]$, then the Fourier series of f is of the form $f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)$.

(iv) The quadratic form $Q(x_1, x_2, x_3) = (x_1 + x_2)^2 + (x_1 + x_3)^2 + 2(x_2 + x_3)^2$ is positive definite.

(v) Let A be an orthogonal matrix. Then, all the singular values of A are 1.

(vi) There are real numbers a, b, such that $y_1(t) = e^t$ and $y_2(t) = \cos(t)$ are both solutions to $y'' + ay' + by = 0$.

(vii) Suppose A is a 2 by 2 diagonalizable matrix with $\det(A) = 0$. Then, the system of differential equations

$$\mathbf{x}' = A\mathbf{x}$$

has a nonzero solution $\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, where x_1 and x_2 are constant functions.

(viii) If A is a 3×3 matrix with $A^2 = 0$, then $\operatorname{rank}(A) \leqslant 1$.

Second page for Question 1.

Question 2. Let
$$W = \text{Span}\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\right) \text{ and } \vec{y} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$
Find the vector $\hat{y} \in W$ that minimizes the quantity $||\hat{y} - \vec{y}||$.

Second page for Question 2.

Question 3. Find the solution to the initial value problem

$$\begin{aligned} x_1'(t) &= x_1(t) - x_2(t) \qquad & x_1(0) &= 1 \\ x_2'(t) &= x_1(t) + x_2(t) \qquad & x_2(0) &= -1 \end{aligned}$$

Second page for Question 3.

Question 4. Find the general solution to the differential equation

$$y' - 3y = e^{3t} - 3e^{3t}\cos(3t)$$

Second page for Question 4.

Question 5. Define $V \subset \mathbb{R}^3$ as $V := \text{Span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}\right)$. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{u}) = \text{proj}_V(\mathbf{u})$.

a) Find an orthonormal basis for V and compute the standard matrix, A, for T.
b) Find a diagonal matrix D and an invertible matrix P, such that $A = PDP^{-1}$.
c) Compute the singular values of A.

Second page for Question 5.

Question 6. For this problem, please fill in each blank with the appropriate words, numbers, or equations that complete the reasoning. If you write your answers on a separate page, **clearly** indicate which answer goes in which blank (or your submission will not be graded).

Suppose A is a real symmetric 4×4 matrix such that $A^2 = A$ and $\text{rank}(A) = 3$.

a) Show that the only possible eigenvalues of A are 0 and 1.

Solution: Suppose that λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector with eigenvalue λ. Then,

1 ____

$$\implies \lambda^2 \mathbf{v} = \lambda \mathbf{v}$$

Therefore, $(\lambda^2 - \lambda)\mathbf{v} = 0$, so because $\mathbf{v}$ is 2 ______________, it follows that $\lambda^2 - \lambda = 0$ and $\lambda = 0$ or 1.

b) Determine the dimension of each eigenspace of A. (In this question, we denote the eigenspace with eigenvalue λ by E_λ.)

Solution: First, $\dim(\text{Nul}(A)) =$ 3 ______ because

4 ____.

The dimension of 5 ______ is equal $\dim(\text{Nul}(A))$. Furthermore, the dimensions of E_0 and E_1 add up to 4 because

6 ____.

Therefore, the dimension of E_0 is 1 and the dimension of E_1 is 3.

c) $\det(A^{2020} + I) =$ 7 ______ (No reasoning required for this blank.)

Second page for Question 6.

Question 7. Consider the function defined on the interval $[-\pi, \pi]$ given by

$$f(x) = \begin{cases} 1, & -\pi \leqslant x \leqslant 0 \\ \cos(x), & 0 < x \leqslant \pi \end{cases}.$$

(a) Compute the four coefficients a_2, a_3, b_2, b_3 in the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right].$$

You may use the following formulas without justification. For any integer $n \geqslant 1$:

$$\int_0^\pi \cos(x)\sin(nx)\,dx = \begin{cases} 0, & n = 1 \\ \frac{((-1)^n+1)n}{n^2-1}, & n \geqslant 2 \end{cases}$$

$$\int_0^\pi \cos(x)\cos(nx)\,dx = \begin{cases} \pi/2, & n = 1 \\ 0 & n \geqslant 2 \end{cases}.$$

(b) What does the Fourier series converge to at $x = \pi$?

Second page for Question 7.