MATH 110 – FINAL EXAM

PROFESSOR TARA S. HOLM

December 16, 2003

Time limit: 3 hours

Name:

SID:

- All answers must be written in complete English sentences. Any solution written grammatically incorrectly will not receive full credit.
- You may not consult any books or papers. You may not use a calculator or any other computing or graphing device other than your own head!
- There are some pages where you must choose one of two problems. Please read all instructions carefully.
- Please write your name on *EVERY* page of this exam.
- There are four extra blank pages at the end of the exam. You may use these for computations, but **I will not read them.** Please transfer all final answers to the page on which the question is posed.

Problem:	Your score:	Total points
1		20 points
2		20 points
3		20 points
4		20 points
5		20 points
6		20 points
7		10 points
8		20 points
9		20 points
10		10 points
11		20 points
Total:		200 points

1. **True/False. (20 points)** Please answer **ONE** of the following **TWO** questions true or false. Justify your answer: give a proof or counterexample. You will not receive extra credit for answering more than one question.

   For this problem, you may want to refer to the Pythagorean theorem, which states that if u and v are orthogonal vectors, then $\|u\|^2 + \|v\|^2 = \|u + v\|^2$.

   (a) **True or False**: Let V be a real inner product space. If $\|u\|^2 + \|v\|^2 = \|u + v\|^2$, then u and v are orthogonal vectors.
   Please circle your answer: | True | False |

   (b) **True or False**: Let V be a complex inner product space. If $\|u\|^2 + \|v\|^2 = \|u + v\|^2$, then u and v are orthogonal vectors.
   Please circle your answer: | True | False |

2. **True/False. (20 points)** Please answer **ONE** of the following **TWO** questions true or false. Justify your answer: give a proof or counterexample. You will not receive extra credit for answering more than one question.

   (a) **True or False**: Let $T : V \to V$ be a linear operator. If λ is an eigenvalue of T, then λ is an eigenvalue of T^2.

   Please circle your answer: | True | False |

   (b) **True or False**: Let $T : V \to V$ be an invertible linear operator. If v is a non-zero eigenvector of T, then v is a non-zero eigenvector of T^{-1}.

   Please circle your answer: | True | False |

3. **True/False. (20 points)** Please answer **ONE** of the following **TWO** questions true or false. Justify your answer: give a proof or counterexample. You will not receive extra credit for answering more than one question.

   (a) **True or False**: If W is a subspace of $\mathbb{R}^3$, then there exists a transformation $T \in \mathcal{L}(\mathbb{R}^3)$ with $R(T) = W$.
   Please circle your answer: | True | False |

   (b) **True or False**: If W is a subspace of $\mathbb{R}^3$, then there exists a transformation $T \in \mathcal{L}(\mathbb{R}^3)$ with $N(T) = W$.
   Please circle your answer: | True | False |

4. **True/False. (20 points)** Please answer **BOTH** of the following **TWO** questions true or false. Justify your answer: give a proof or counterexample.

   (a) **True or False**: The zero transformation on $\mathbb{C}^2$ has infinitely many square roots.
   Please circle your answer:

True	False

   (b) **True or False**: Suppose that U and W are both 4-dimensional subspaces of a 6-dimensional vector space V. Then $\dim(U \cap V) = 2$.
   Please circle your answer:

True	False

5. **(20 points)** Let $S : U \to V$ and $T : V \to W$ be linear transformations.

(a) **Prove** that $\dim(R(S \circ T)) \leq \dim(R(S))$.

(b) **Find** and example where $\dim(R(S \circ T)) < \dim(R(S))$.

6. **(20 points)** Equip $\mathbb{R}^3$ with the standard inner product (the dot product), and consider the subspace

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}.$$

(a) Find a basis for W.

(b) Find a basis for $W^{\perp}$.

7. **(10 points)** Equip $\mathbb{R}^3$ with the standard inner product. Consider the vectors $v_1 = (1, -1, -1)$, $v_2 = (0, 3, 3)$, and $v_3 = (3, 2, 4)$. Find an orthonormal list of vectors (u_1, u_2, u_3) so that $span(u_1) = span(v_1)$, $span(u_1, u_2) = span(v_1, v_2)$, and $span(u_1, u_2, u_3) = span(v_1, v_2, v_3)$.

8. **(20 points)** Consider the operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + 3z, 2y, -3x + z)$.

(a) Is T self-adjoint? Why or why not?

(b) Is T normal? Why or why not?

9. **(20 points)** Consider the operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by

$$T(x, y, z) = (-4x + 7y, 3y, 2x + 3y - 2z).$$

One eigenvalue of this operator is 3.

(a) What are the second and third eigenvalues of T?

(b) Find a non-zero eigenvector for **ONE** of the eigenvalues you computed in (a).

10. **(10 points)** Let $A \in M_{n\times n}(\mathbb{R})$. Suppose that $A^{\mathrm{T}} = A$ (where $^{\mathrm{T}}$ denotes transpose). Prove that there exist matrices Q and D, with D a diagonal matrix, so that

$$A = QDQ^{-1}.$$

11. **(20 points)** Let V be a finite-dimensional vector space, and fix an invertible transformation $S \in \mathcal{L}(V)$. Define a function

$$\Psi_S : \mathcal{L}(V) \to \mathcal{L}(V)$$

by $\Psi_S(T) = S \circ T \circ S^{-1}$.

(a) Prove that Ψ_S is a linear transformation.

(b) Compute $N(\Psi_S)$.

(c) Prove that if λ is an eigenvalue of T, then λ is an eigenvalue of $\Psi_S(T)$.