Combinatorial and set-theoretic forcing

Stevo Todorcevic

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Baire's representation theorem

Theorem (Baire, 1899)

A real function g on a Polish space X is a pointwise limit of a sequence of continuous functions on X if and only if g has a point of continuity on any nonempty closed subset of X.

A real function g on a Polish space X is of **Baire-class-1** if g is a point wise limit of a sequence of continuous functions on X.

Let $\mathcal{B}_1(X)$ be the collection of **Baire-class-1 functions** on a given Polish space X.

$\mathcal{B}_1(X)$ comes with the **topology of pointwise convergence on** X, i.e., the subspace topology induced from the Tychonoff cube $\mathbb{R}^X$.

Compact sets of Baire-class-1 functions

Theorem (Odell-Rosenthal, 1975)

Suppose that a separable Banach space X contains no isomorphic copy of the space ℓ_1.
Then the unit ball $B_{X^{**}}$ of the double-dual X^{**} of X considered as a collection of functions on the dual ball B_{X^*} equipped with the weak*-topology consists of Baire-class-1 functions on B_{X^*}.
Moreover, every x^{**} from $B_{X^{**}}$ is a weak**-limit (i.e., pointwise limit) of a sequence $(x_n) \subseteq B_X$.

Thus if $\ell_1 \not\subseteq X$, the ball $B_{X^{**}}$ is a **compact convex set** of Baire-class-1 functions on B_{X^*}.

Problem

Which compact are representable inside the space of Baire-class-1 functions on $\mathbb{N}^{\mathbb{N}}$?

Helly's space and the split interval $[0,1] \times \{0,1\}$

Helly space is the compact convex set H of **all monotone mappings** from $[0,1]$ to $[0,1]$.

The **extremal points** of the **Helly space** H is the set of all monotone mappings from $[0,1]$ to $\{0,1\}$ and is homeomorphic to the **split interval** $[0,1] \times \{0,1\}$.

Thus, a representation of **the split interval** $[0,1] \times \{0,1\}$ as a subspace of $\mathcal{B}_1([0,1])$ is given by:

$$(x,0) \mapsto \chi_{[0,x)} \text{ and } (x,1) \mapsto \chi_{[0,x]}.$$

Cantor tree space: Pol's compactum

For a Polish space X,

$$A(X) = \{\chi_{\{x\}} : x \in X\} \cup \{\chi_\emptyset\}$$

is a representation of the **one-point compactification** of the discrete space $\{\chi_{\{x\}} : x \in X\}$.
For $X = 2^{\mathbb{N}}$ and $s \in 2^{<\mathbb{N}}$ let χ_s denote the characteristic function of the corresponding basic clopen subset of $2^{\mathbb{N}}$. Let

$$PA(2^{\mathbb{N}}) = A(2^{\mathbb{N}}) \cup \{\chi_s : s \in 2^{<\mathbb{N}}\}$$

is the **Pol compactum**, a separable compact subspace of $\mathcal{B}_1(2^{\mathbb{N}})$ with $\chi_\emptyset$ as a **non-G_δ-point**, **the point at infinity**.

James tree space

Theorem (James, 1974)

There is a separable Banach space JT such that JT^* is not separable but JT contains no subspace isomorphic to ℓ_1.

Thus, the **double-dual ball** of JT is a compact convex set of Baire-class-1 functions with 0^{**} as a **non G_δ-point.**

In fact, **the weak*-closure** of the basis of JT in the double-dual ball of JT is naturally homeomorphic to the **Pol space** $PA(^{\mathbb{N}})$. More precisely, there is a homoeomorphic embedding

$$\Phi : PA(2^{\mathbb{N}}) \to B_{JT^{**}}$$

such that $\Phi(\infty) = 0^{**}$ and

$$\Phi[2^{<\mathbb{N}}] = \text{the basis of } JT.$$

The Alexandroff duplicate of $2^{\mathbb{N}}$

The **Alexandroff duplicate** $D(2^{\mathbb{N}}) = 2^{\mathbb{N}} \times \{0, 1\}$ is represented inside the first Baire class via the mapping $\Phi : D(2^{\mathbb{N}}) \to 2^{\mathbb{N}} \times A(2^{\mathbb{N}})$:

$$\Phi(x, 0) = (x, \chi_{\emptyset}) \text{ and } \Phi(x, 1) = (x, \chi_{\{x\}}).$$

This copy of the Alexandroff duplicate $D(2^{\mathbb{N}})$ could also be supplemented to the **separable version of the Alexandroff duplicate** $SD(2^{\mathbb{N}})$ by adding to the image of Φ the countable dense set

$$\{(s^\frown 0^{(\omega)}, \chi_s) : s \in 2^{<\mathbb{N}}\}.$$

Baire-class-1 compacta

Let us say that a compact space K is a **Baire-class-1 compactum** if it is homeomorphic to a compact subset of $\mathcal{B}_1(X)$ for some Polish space X.

Problem

Which chain conditions are identified in the calss of all Baire-class-1 compact?

More concretely, we can ask the following version of the **Souslin Problem**

Problem

Suppose that a Baire-class-1 compactum K satisfies the **countable chain condition**. *Is K necessarily* **separable**?

Dense metrizable subspaces

Theorem (T., 1999)

Every Baire-class-1 compactum has a dense metrizable subspace.

Corollary (T., 1999)

Every Baire-class-1 compactum that satisfies the **countable chain condition** *is* **separable**.

Corollary (Bourgain, 1978)

Every Baire-class-1 compactum has a dense set of G_δ *points.*

Problem (Bourgain, 1978)

Is the set of all G_δ *points in a Baire-class-1 compactum* K *a comeager subset of* K*?*

Forcing and Baire-class-1 compacta

Fix an arbitrary poset $\mathbb{P}$ and consider it as a **forcing notion**.
We may (and will) restrict to the Baire class $\mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$.
In the **forcing extension** of $\mathbb{P}$ the Polish space $\mathbb{N}^{\mathbb{N}}$ has its natural interpretation which we denote by $\hat{\mathbb{N}^{\mathbb{N}}}$.
Similarly, in the **forcing extension** of $\mathbb{P}$, a **continuous** real function f on $\mathbb{N}^{\mathbb{N}}$ extends to a **continuous** real function $\hat{f}$ on $\hat{\mathbb{N}^{\mathbb{N}}}$.

Lemma

If (f_n) is a pointwise-convergent sequence of continuous real functions on $\mathbb{N}^{\mathbb{N}}$ then $\mathbb{P}$ forces that the corresponding sequence $(\hat{f}_n)$ is pointwise convergent on $\hat{\mathbb{N}^{\mathbb{N}}}$.
Moreover, if (g_n) is another pointwise-convergent sequence of continuous real functions on $\mathbb{N}^{\mathbb{N}}$ converging to the same limit then $\mathbb{P}$ forces that $(\hat{f}_n)$ and $(\hat{g}_n)$ converge to the same limit.

Thus, every Baire-class-one function h on $\mathbb{N}^{\mathbb{N}}$, in the forcing extension of $\mathbb{P}$, extends naturally to a Baire-calss-1 function $\hat{h}$ on $\hat{\mathbb{N}^{\mathbb{N}}}$.

Theorem (T., 1999)

If K is a **relatively compact** *subset of $\mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$ then $\mathbb{P}$ forces that*

$$\hat{K} = \{\hat{f} : f \in K\}$$

is a **relatively compact** *subset of $\mathcal{B}_1(\hat{\mathbb{N}^{\mathbb{N}}})$.*

Corollary (Bourgain, 1984)

For every Radon measure μ on a Baire-class-1 compactum K the space $L^1(K, \mu)$ is separable.

Proof.

If μ is not separable then by Fremlin's theorem (second lecture) there is a poset $\mathbb{P}$ satisfying the **countable chain condition** which forces that the closure of $\hat{K}$ maps onto the Tychonoff cube $[0, 1]^{\omega_1}$. We will see that no Baire-class-1 compactum can map onto $[0, 1]^{\omega_1}$.

□

Fix a Baire-class-1 compactum $K \subseteq \mathcal{B}_1(\mathbb{N}^{\mathbb{N}})$. Let $\mathbb{B}(K)$ be the complete Boolean algebra of **regular-open** subsets of K and let

$$\mathbb{P}(K) = \mathbb{B}(K) \setminus \{\emptyset\}.$$

Lemma

$\mathbb{P}(K)$ *forces that its generic filter is* **countably generated**

This uses a particular form of point-countable π-basis of K mentioned above in the second lecture.

Corollary

Every Baire-class-1 compactum has a σ-disjoint π-basis.

Convergence in $\mathcal{B}_1(X)$

Theorem (Rosenthal, 1977)

If K is a Baire-class-1 compactum then every sequence $(f_n) \subseteq K$ has a convergent subsequence (f_{n_k}).

In other words, every Baire-class-1 compactum is **sequentially compact**.

Theorem (Rosenthal, 1977)

Every Baire-class-1 compactum is **countably tight.**

Theorem (Bourgain-Fremlin-Talagrand, 1978)

Every Baire-class-1 compactum is a **Fréchet space.**

Theorem (Bourgain-Fremlin-Talagrand, 1978)

Suppose that K is a compact subset of $\mathcal{B}_1(X)$ for some Polish space X. Then $\overline{\mathrm{conv}}(K)$ taken in $\mathbb{R}^X$ is included in $\mathcal{B}_1(X)$.

Split interval again

Note that the projection

$$\pi_1 : [0,1] \times \{0,1\} \to [0,1]$$

is a 2-to-1 continuous map from the **split interval** $[0,1] \times \{0,1\}$ onto the unit interval $[0,1]$.
Note also that **every closed subspace** of $[0,1] \times \{0,1\}$ satisfies the **countable chain condition**.
In fact, **every closed subset** of $[0,1] \times \{0,1\}$ is **separable and** G_δ.

Theorem (T., 1999)

If K is a Baire-class-1 compactum then either

1. *K contains a closed subspace that* **fails the countable chain condition**, *or*
2. *There is a continuous map $f : K \to M$ from K onto some metric space M such that that $|f^{-1}(x)| \leq 2$ for all $x \in M$.*

Theorem (T., 1999)

Let K be a Baire-class-1 compactum. Then

1. *K is* **metrizable**, *or*
2. *K contains a closed subspace* **failing the countable chain condition**, *or*
3. *K contains a homeomorphic copy of the* **split interval** $[0,1] \times \{0,1\}$.

Corollary

Suppose K is a Baire-class-1 compactum in which **all closed subsets are** G_δ *and that K contains* **no copy of the split interval**. *Then K is either* **metrizable**

The duplicate of the Cantor space

Recall that

$$D(2^{\mathbb{N}}) = 2^{\mathbb{N}} \times \{0,1\}$$

is the **Alexandroff duplicate of the Cantor space** $2^{\mathbb{N}}$ with points $(x,1)$ $(x \in 2^{\mathbb{N}})$ isolated.

The **separable Alexandroff duplicate**

$$SD(2^{\mathbb{N}}) = 2^{<\mathbb{N}} \cup D(2^{\mathbb{N}})$$

is obtained by adding $2^{<\mathbb{N}}$ as a dense set of isolated points.

Theorem (T., 1999)

Suppose that K is a separable Baire-class-1 compactum that admits a continuous function $f : K \to M$ onto a metric space M such that $|f^{-1}(x)| \leq 2$ for all $x \in M$. Then at least one of the following three alternatives must hold:

1. *K is metrizable.*
2. *K contains the separable duplicate $SD(2^{\mathbb{N}})$.*
3. *K contains the split interval $[0,1] \times \{0,1\}$.*

Forbidding the duplicate $D(2^{\mathbb{N}})$

Theorem (T., 1999)

Suppose that K is a Baire-class-1 compactum that admits a continuous function $f : K \to M$ onto a metric space M such that $|f^{-1}(x)| \leq 2$ for all $x \in M$.
Then the following three conditions are equivalent:

1. *Every closed subspace of K satisfies the* **countable chain condition**.
2. *Every closed subset of K is G_δ in K.*
3. *K contains no copy of $D(2^{\mathbb{N}})$.*

Points in Baire-class-1 compacta

Recall the **Pol compactum**

$$PA(2^{\mathbb{N}}) = 2^{<\mathbb{N}} \cup 2^{\mathbb{N}} \cup \{\infty\},$$

the one-point compactification of the **Cantor tree space**

$$2^{<\mathbb{N}} \cup 2^{\mathbb{N}},$$

the locally compact space generated by the complete binary tree $2^{<\mathbb{N}}$ with $2^{\mathbb{N}}$ as a set of its branches, where

1. points of the tree $2^{<\mathbb{N}}$ are isolated, and where
2. basic-open neighbourhoods of a branch $x \in 2^{\mathbb{N}}$ are its tails

$$\{x \upharpoonright n : n \geq m\} \cup \{x\}$$

Note that

$$PA(2^{\mathbb{N}}) \setminus 2^{<\mathbb{N}} = A(2^{\mathbb{N}}),$$

the one-point compactification of a discrete space of cardinality continuum so the point at infinity is **not a G_δ-point** .

Theorem (T., 1999)

Suppose that K is a separable Baire-class-1 compactum, that z is its **non-G_δ-point***, and that D is its countable dense subset of K. Then:*

1. *K contains a copy of $PA(2^{\mathbb{N}})$ with z as its point at infinity and its countable dense set included in D.*
2. *Moreover, the embedding $\Phi : PA(2^{\mathbb{N}}) \to K$ is given by a Borel map $\Psi : 2^{\mathbb{N}} \to \mathcal{B}_1(X)$.*

Ramsey methods

Theorem (T., 1999)

Let f_s $(s \in 2^{<\mathbb{N}})$ *be a* **relatively compact** *subset of* $\mathcal{B}_1(X)$ *for some Polish space* X. *Then there is a perfect set* $P \subseteq 2^{\mathbb{N}}$ *and an infinite strictly increasing sequence* (n_k) *of integers such that for every* $a \in 2^{\mathbb{N}}$ *the sequence* $(f_{a\restriction n_k})$ **pointwise converges on** X *and, if we let* f_a *denote its limit,*

$$\Psi(a, x) = f_a(x)$$

defines a Borel function from $P \times X$ *into* $\mathbb{R}$.

This opened the possibility of using the theory of **Ramsey spaces** into the study of subsets of $\mathcal{B}_1(X)$ and therefore the study of Banach spaces containing no ℓ_1.
Particularly important in this case is the **Ramsey space of trees** that is based on the **Halpern-Läuchli theorem.**

Separable quotient problem

Theorem (Mazur, 1930)

Every infinite dimensional Banach space contains an infinite dimensional **subspace with a basis.**

Problem (Banach 1930: Pelczynski 1964)

Does every infinite dimensional Banach space has an infinite dimensional **quotient with a basis?**

Theorem (Johnson-Rosenthal, 1972)

Every **separable** *infinite dimensional Banach space has an infinite dimensional quotient with a basis.*

Problem (Jphnson-Rosenthal, 1972)

Does every infinite dimensional Banach space has an infinite dimensional **separable quotient?**

Unconditional sequences

Definition

A sequence x_i $(i \in I)$ of points in some Banach space is **unconditional** if it is normalized and if we can find a constant $C \geq 1$ such that for every pair $G \subseteq H$ of finite subsets of the index-set I and for every choice of scalars λ_i $(i \in H)$, we have

$$\|\sum_{i \in G} \lambda_i x_i\| \leq C \|\sum_{i \in H} \lambda_i x_i\|.$$

Theorem (Johnson-Rosenthal 1972; Hagler-Johnson, 1977)

If the dual X^* of some Banach space X contains an **infinite unconditional sequence** *then X has an infinite dimensional* **quotient with a basis.**

Mycielski independence theorem

Theorem (Mycielski, 1964)

Suppose X is a perfect Polish space and that for each positive integer n we are given a **meager** *subset M_n of X^n. Then there is a* **perfect subset** *P of X such $P^{(n)} \cap M_n = \emptyset$ that for every n, where*

$$P^{(n)} = \{(x_1, ..., x_n) \in P^n : x_i \neq x_j \text{ for all } i \neq j\}.$$

Lemma (Argyros-Dodos-Kanellopoulos, 2008)

Suppose that X is a Polish space and that $\Psi : 2^{\mathbb{N}} \times X \to \mathbb{R}$ is a Borel function such that:

1. *the sequence $f_a = \Psi(a, \cdot)$ $(a \in 2^{\mathbb{N}})$ is bounded in $\ell_\infty(X)$,*
2. *the set $\{a \in 2^{\mathbb{N}} : f_a(x) \neq 0\}$ is countable for all $x \in X$.*

Then there is a perfect set $P \subseteq 2^{\mathbb{N}}$ such that the sequence f_a $(a \in P)$ is unconditional in $\ell_\infty(X)$.

Theorem (Argyros-Dodos-Kanellopoulos, 2008)

Every infinite dimensional **dual Banach space** X^* *has an infinite dimensional* **quotient with a basis.**

Proof.

We may assume that X^* is not separable and that the predial X is separable and that it contains no ℓ_1.
Then the unit ball $B_{X^{**}}$ is a separable Baire-class-1 comactum with 0^{**} as its **non-G_δ-point**.
By the structure theorem there is an embedding

$$\Phi : PA(2^{\mathbb{N}}) \to B_{X^{**}}$$

such that $\Phi(\infty) = 0^{**}$ and such that Φ is given by a Borel map

$$\Psi : 2^{\mathbb{N}} \times B_{X^*} \to \mathbb{R}.$$

Note that the hypotheses of the lemma are satisfied .
So X^{**} has an infinite unconditional sequence and so X^* has the required quotient. □

Classifying families of sequences

Definition

A **co-ideal** on $\mathbb{N}$ is a family $\mathcal{H}$ of infinite subsets of $\mathbb{N}$ such that

1. if $M \subseteq N$ and if $M \in \mathcal{H}$ then $N \in \mathcal{H}$.
2. if $M \in \mathcal{H}$ and $M = M_0 \cup M_1$ then either $M_0 \in \mathcal{H}$ or $M_1 \in \mathcal{H}$.

A co-ideal $\mathcal{H}$ on $\mathbb{N}$ is **selective** if for every $M \in \mathcal{H}$ and every $f : M \to \mathbb{N}$ there is $N \in \mathcal{H}$, $N \subseteq M$ such that $f \restriction N$ is either **constant** or **one-to-one.**

Example

The co-ideal of **infinite subsets** of $\mathbb{N}$ is selective.

Theorem (Mathias, 1977)

A co-ideal $\mathcal{H}$ is **selective** *if and only if for every finite* **Souslin-measurable** *colouring of the collection $\mathbb{N}^{[\infty]}$ of all infinite subsets of $\mathbb{N}$ there is $M \in \mathcal{H}$ such that $M^{[\infty]}$ is monochromatic.*

Fix a sequence (x_n) in a Baire-class-1 compactum K and fix a point $x \in K \setminus \{x_n : n \in \mathbb{N}\}$. Let

$$\mathcal{H}_K(x,(x_n)) = \{M \subseteq \mathbb{N} : x \in \overline{\{x_n : n \in M\}}\}.$$

Theorem (T. 1995)

$\mathcal{H}_K(x,(x_n))$ is a **selective co-ideal**, *or equivalently, for every finite* **Souslin-measurable** *colouring of the collection $\mathbb{N}^{[\infty]}$ of all infinite subsets of $\mathbb{N}$ there is $M \in \mathcal{H}_K(x,(x_n))$ such that $M^{[\infty]}$ is monochromatic.*

Corollary (Bourgain-Fremlin-Talagrand, 1978)

Every Baire-class-1 compactum is a **Fréchet space**.

Proof.

Color a subset N of $\mathbb{N}$ **blue** if the sequence $(x_n)_{n \in N}$ converges to x; otherwise, colour N **red**. □

Gaps

Let

$$\mathcal{C}_K(x,(x_n)) = \{M \subseteq \mathbb{N} : (x_n)_{n\in M} \text{ converges to } x\}$$

and

$$\mathcal{D}_K(x,(x_n)) = \{M \subseteq \mathbb{N} : x \notin \overline{\{x_n : n \in M\}}\}.$$

Lemma

$\mathcal{C}_K(x,(x_n))$ and $\mathcal{C}_K(x,(x_n))$ are two **orthogonal** *families of subsets of $\mathbb{N}$ that* **can't be separated** *unless x is an isolated point in K.*

Problem

What is the structure of gaps $(\mathcal{C},\mathcal{D})$ of this form?

Remark

Recall that the **Von Neumann-Maharam problem** asks the same for the gap formed by the family of sequences in $\mathbb{B}^+$ that **converge to** 0 and the family of sequences that **do not accumulate to** 0, where $\mathbb{B}$ is a complete Boolean algebra satisfying the **countable chain condition** and the **weak countable distributive law**.

A canonical gap

Consider the following gap on the complete binary tree $2^{<\mathbb{N}}$

$$\mathcal{A}_0(2^{<\mathbb{N}}) = \{M \subseteq 2^{<\mathbb{N}} : M \text{ is an infinite antichain}\},$$

$$\mathcal{A}_1(2^{<\mathbb{N}}) = \{M \subseteq 2^{<\mathbb{N}} : M \text{ conains no infinite antichain}\}.$$

Note that this is really the gap of the form

$$(\mathcal{C}_K(x,(x_n)), \mathcal{D}_K(x,(x_n))),$$

where K is the Pol compactum $PA(2^{\mathbb{N}})$.

Theorem (T., 1999)

Every gap of the form $(\mathcal{C}_K(x,(x_n)), \mathcal{D}_K(x,(x_n)))$, where K is a Baire-class-1 compactum and x its **non-G_δ-point**, *has a restriction isomorphic to the gap $(\mathcal{A}_0(2^{<\mathbb{N}}), \mathcal{A}_1(2^{<\mathbb{N}}))$.*

General theory of gaps

Definition

A **preideal** on a countable index-set N is a family I of infinite subsets of N such that if $x \in I$ and $y \subseteq x$ is infinite, then $y \in I$.

Definition

Let $\Gamma = \{\Gamma_i : i \in n\}$ be a n-sequence of preideals on a set N and let $\mathfrak{X}$ be a family of subsets of n.

1. We say that Γ is **separated** if there exist subsets $a_0, \ldots, a_{n-1} \subseteq N$ such that $\bigcap_{i \in n} a_i = \emptyset$ and $x \subseteq^* a_i$ for all $x \in \Gamma_i$, $i \in n$.
2. We say that Γ is an $\mathfrak{X}$-**gap** if it is not separated, but $\bigcap_{i \in A} x_i =^* \emptyset$ whenever $x_i \in \Gamma_i$, $A \in \mathfrak{X}$.

Definition

When $\mathfrak{X}$ is the family of all subsets of n of cardinality 2, an $\mathfrak{X}$-gap will be called an n-**gap**,
When $\mathfrak{X}$ consists only of the total set $\{0, \ldots, n-1\}$, then an $\mathfrak{X}$-gap will be called an n_*-**gap**.

Definition

The **orthogonal**, $I^{\perp}$, of a preideal I on N is the family of all infinite subsets of N that have **finite intersections** with all sets from I.

Definition

For Γ and Δ two n_*-gaps on countable sets N and M, respectively, we say that

$$\Gamma \leq \Delta$$

if there exists a one-to-one map $\phi : N \to M$ such that for $i < n$,

1. if $x \in \Gamma_i$ then $\phi(x) \in \Delta_i$.
2. If $x \in \Gamma_i^{\perp}$ then $\phi(x) \in \Delta_i^{\perp}$.

Definition

Two n_*-gaps Γ and Γ' are called **equivalent** if $\Gamma \leq \Gamma'$ and if $\Gamma' \leq \Gamma$.

A Finite Basis Theorem

Definition

An $_*$-gap Γ is **analytic** if all the preideals of Γ are **analytic families of subsets** of the countable index set N.

Definition

An analytic n_*-gap Γ is said to be a **minimal analytic n_*-gap** if for every other analytic n_*-gap Δ, if $\Delta \leq \Gamma$, then $\Gamma \leq \Delta$.

Theorem (Aviles-Todorcevic, 2013)

Fix a natural number n. For every analytic n_*-gap Γ there exists a minimal analytic n_*-gap Δ such that $\Delta \leq \Gamma$. Moreover, up to equivalence, there exist only finitely many minimal analytic n_*-gaps.

Remark

Up to permutations there exist exactly
5 **minimal analytic** 2-**gaps** (9 in total) and
163 **minimal analytic** 3-**gaps** (933 in total).

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