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Tarski Lecture 2 in unicode

1 message

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Lecture 2. Reverse Mathematics: A Global View

April 26, 4:10pm, Evans 60

We view the structure of reverse mathematics as a degree structure similar to that of the Turing degrees, $\mathcal{D}_T$ with the ordering of Turing reducibility, $\leq_T$. We define an ordering $\leq_P$ on sets of sentences of second order arithmetic $S \leq_P T \Leftrightarrow RCA_0 \cup T \vdash \varphi$ for every $\varphi \in S$. As usual we consider the induced ordering $\leq_P$ on the equivalence classes $\boldsymbol{s}$ and $\boldsymbol{t}$ and the resulting structure $\mathcal{D}_P$. Important substructures are $\mathscr{F}_P$ and $\mathscr{R}_P$ consisting of the degrees of finitely and recursively axiomatizable sets of sentences. We prove a large number of global results about $\mathcal{D}_P$ that differ remarkably from those for the analogous questions about $\mathcal{D}_T$ and other degree structures. A few sample results are the following.

Theorem: $\mathcal{D}_P$ is a complete algebraic lattice (with 0 and 1) and so pseudocomplemented. $\mathscr{R}_P$ is an incomplete algebraic lattice. For each of them the compact elements are those in $\mathscr{F}_P$ and the pseudocomplement of T in $\mathcal{D}_P$ is $\vee\{\varphi \mid \varphi \wedge T = 0\}$. $\mathscr{F}_P$ is the atomless Boolean algebra.

Theorem: The (first order) theories of $\mathscr{F}_P$ and $\mathcal{D}_P$ with $\leq_P$ (and so with $\vee$, $\wedge$, 0, and 1) are decidable by applying major results of Tarski and Rabin.

Theorem: $\mathcal{D}_P$ and $\mathscr{F}_P$ have exactly 2^ω many automorphisms

Theorem: There are only four finite sets which are definable in $\mathcal{D}_P$: $\emptyset$, $\{0\}$, $\{1\}$ and $\{0,1\}$. There are only four countably infinite definable subsets of $\mathcal{D}_P$: $\mathscr{F}_P$, $\mathscr{F}_P$ -

{0}, $\mathscr{F}_P$ - *{1}* and $\mathscr{F}_P$ - *{0,1}*. In each case, no other such sets are fixed under all automorphisms of $\mathcal{D}_P$.

Theorem: Up to isomorphism, there are only four cones $\mathcal{D}_P{}^s = \{\boldsymbol{t} \mid \boldsymbol{s} \leq_P \boldsymbol{t}\}$ in $\mathcal{D}_P$: $\{\boldsymbol{1}\}$, $\{\boldsymbol{s}, \boldsymbol{1}\}$, $\{\boldsymbol{s}, \boldsymbol{s} \vee \boldsymbol{t}_0, \boldsymbol{s} \vee \boldsymbol{t}_1, \boldsymbol{1}\}$ and $\mathcal{D}_P$. We can characterize the few ***s*** that fall into each of the first three classes in terms of notions familiar in the general study of theories.

This analysis was prompted by my thinking about what I should talk about in these lectures. Much to my surprise, after I had worked out most of these results I discovered that Tarski had proven many of them some ninety years ago and so long before the rise of reverse mathematics.