George M. Bergman
5 Evans Hall

Spring 1994, Math 114
Final Exam

20 May, 1994
3 hours, between 4 and 8 PM

1. (40 points) Mark statements *T* (true) or *F* (false). Each correct answer will count 1 point, each incorrect answer −1 point, each unanswered item 0 points.

____ $\mathbf{Z}_n$ is an integral domain if and only if it is a field.

____ Every polynomial over a field K has a root in K.

____ Every polynomial of prime degree is irreducible.

____ All simple transcendental extensions of a given field are isomorphic.

____ All simple algebraic extensions of a given field are isomorphic.

____ If $K(\alpha) \cong K(\beta)$ as extension-fields of K, then α and β have the same minimal polynomial.

____ Given a line-segment of length 1, one can construct by ruler and compass a line-segment of length $3^{1/4}$.

____ If L is a field, the only L-automorphism of L is the identity.

____ If $S \subseteq \mathbf{R}^2$ is the circle of radius 1 centered at the origin, then for every point $(\alpha, \beta) \in S$, the field $\mathbf{Q}(\alpha, \beta)$ is algebraic over $\mathbf{Q}$, of degree a power of 2.

____ The Galois group $\Gamma(\mathbf{C}:\mathbf{R})$ is abelian.

____ Every separable field extension is normal.

____ If L is generated over K by elements separable over K, then every element of L is separable over K.

____ If K and L are fields of the same characteristic, then there exists a homomorphism $K \to L$.

____ If K and L are fields and there exists a homomorphism $K \to L$, then K and L have the same characteristic.

____ Distinct automorphisms of a field K are linearly independent over K.

____ The extension $\mathbf{Q}(2^{1/2}):\mathbf{Q}$ is normal.

____ The extension $\mathbf{Q}(2^{1/3}):\mathbf{Q}$ is normal.

____ The extension $\mathbf{Q}(2^{1/4}):\mathbf{Q}$ is normal.

☞ In the next six parts, let $K \subseteq L \subseteq E$, with E finite over K. ☜

____ If $L:K$ and $E:L$ are both normal, then $E:K$ is normal.

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____ If $E:K$ is normal, then $L:K$ is normal.

____ If $L:K$ and $E:L$ are both separable, then $E:K$ is separable.

____ If $E:K$ is separable, then $E:L$ is separable.

____ If $E:K$ is separable, then $L:K$ is separable.

____ Every group of order 96 has a subgroup of order 8.

____ If G is a group, p is a prime, and H_1, H_2 are p-subgroups of G, then the subgroup of G generated by H_1 and H_2 is also a p-subgroup.

____ If G is a group and N_1, N_2 are normal groups of G, then the subgroup of G generated by N_1 and N_2 is also normal.

____ Every simple solvable group is cyclic.

____ Every finite nontrivial p-group has nontrivial center.

____ Every reducible quintic polynomial over a field of characteristic 0 can be solved by radicals.

____ If $K \subseteq L$ is a finite extension, and tr.deg.$(L{:}K) = 0$, then L is algebraic over K.

____ Every finitely generated field extension is algebraic.

____ There exists a field with 99 elements.

____ The regular 32-gon is constructible with ruler and compass.

____ The regular 33-gon is constructible with ruler and compass.

____ The regular 34-gon is constructible with ruler and compass.

____ If the discriminant of a cubic polynomial $f \in K[t]$ is a square in K, then f is reducible.

____ Every algebraic extension of the field $\mathbf{R}$ of real numbers is normal.

____ For every prime p there exists an ordered field of characteristic p.

____ Every field isomorphic to the field $\mathbf{C}$ of complex numbers is algebraically closed.

2. (45 points) *Definitions and examples.* In this question, when you give an example you do *not* have to prove that your example has the properties asked for.

(a) (10 points) Define what is meant by a *solvable* group, and give examples of a solvable and a nonsolvable group.

(b) (15 points) Define what is meant by the *transcendence degree* of a finitely generated extension $L{:}K$, and for every positive integer n, give an example of a finitely generated extension of transcendence degree n. (In giving the definition, you may assume the concept of algebraically independent family of elements already to have been defined, but not the concept of transcendence basis.)

(c) (10 points) Define what is meant by an *algebraically closed* field, and give an example of an algebraically closed field, and an example of a field that is not algebraically closed.

(d) (10 points) Define what it means for a group G of permutations of a set X to be *transitive*. For some set X, give both a transitive group of permutations of X and a nontransitive group of permutations of X.

3. (50 points) Suppose $L{:}K$ is a finite separable normal extension, whose Galois group is cyclic, with cyclic generator τ of order n; thus the *norm* map $N\colon L \to K$ is given by $N(a) = a\tau(a)\dots\tau^{n-1}(a)$.

(a) (5 points) Show that if $a = b/\tau(b)$ for some $b \in L - \{0\}$, then $N(a) = 1$.

(b) (25 points) Prove the converse statement: if $a \in L$ satisfies $N(a) = 1$, then there exists $b \in L - \{0\}$ such that $a = b/\tau(b)$ (Hilbert's Theorem 90). (If you remember the proof in the book and want to give that, fine. In case you don't, I will remind you of the key idea of the version of that proof I showed in class: consider the properties of the K-linear map $\theta\colon L \to L$ given by $\theta(b) = a\tau(b)$.)

(c) (10 points) Describe briefly the role of Hilbert's Theorem 90 in the proof that a polynomial over a field of characteristic 0 is solvable in radicals if and only if it has solvable Galois group. (E.g., is it used in proving the "if" or the "only if" direction? To what case of the desired result does one apply the Theorem?)

(d) (10 points) Assuming the general hypotheses on $L{:}K$ given at the beginning of this question, suppose that char $K \neq 2$, that L contains a root of $t^2 - 2$, which we will denote $\sqrt{2}$, and that $\tau(\sqrt{2}) = -\sqrt{2}$. Show that if τ has order 2, there does not exist a nonzero element $b \in L$ such that $\tau(b) = (1+\sqrt{2})b$, but that if τ has order 4, there does exist such an element.

4. (15 points) We have seen that if K is a field of prime characteristic p, then the map $a \mapsto a^p$ is an endomorphism of K (meaning a homomorphism of K into itself; in this case called the "Frobenius endomorphism"). In this problem we shall see that "such things only happen in prime characteristic".

(a) (7 points) Show that for any field K, if $f \in K[t]$ and the map $a \mapsto f(a)$ gives an endomorphism of K, then either $f = t$, or the fixed field of this endomorphism is finite. (You may take for granted that the fixed set of an endomorphism of a field *is* a subfield.)

(b) (8 points) Deduce that in the above situation, if $f \neq t$, then K has prime characteristic.