MATH 114

PROFESSOR KENNETH A. RIBET

First Midterm Examination

February 19, 2004

3:40–5:00 PM

Name:

SID:

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

Problem:	Your score:	Total points
1		6 points
2		8 points
3		9 points
4		7 points
Total:		30 points

1. Let n be a positive integer and let p be a prime number. Suppose that x is an integer with $\gcd(x, n) = 1$. Show that there is an integer y such that $\gcd(y, pn) = 1$ and such that $y \equiv x \bmod n$.

Find such a y if $n = 15$, $p = 7$ and $x = 7$.

2. Let p be a prime number. Prove that the polynomial

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^p}{p!}$$

is irreducible over $\mathbb{Q}$.

Prove that $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Z}/2\mathbb{Z}$ (the field of integers mod 2) and that $1234567x^4 - 98765x^3 + 357x^2 - x + 17$ is irreducible over $\mathbb{Q}$.

3. Let $\alpha \approx -2.9196$ be the real root of the polynomial $f(x) := x^3 + 2x^2 - 2x + 2$. Write $\dfrac{1}{\alpha + 3}$ as a polynomial in α.

Let β be a complex root of the polynomial $f(x-7) = x^3 - 19x^2 + 117x - 229$. Show that the fields $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic.

4. For the cubic polynomial $y^3 + py + q = 0$, Cardano's formula reads

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

When $p = -1$ and $q = 0$, the polynomial is $y^3 - y$, which you can solve without the formula (I hope!). Exhibit choices of roots in the formula that lead to the three values -1, 0 and $+1$ for the expression

$$\sqrt[3]{\sqrt{\frac{-1}{27}}} + \sqrt[3]{-\sqrt{\frac{-1}{27}}}.$$