Math 113 Midterm #2, 4/10/03, 8:00 – 9:30 AM M. Hutchings

NAME ______________________________ Score ____________

To receive full credit you must **justify all answers** except where otherwise stated. The point is to demonstrate that you understand the material. No books, notes, calculators, collaboration, or other aids are permitted. There are 5 questions, each with two parts worth 5 points each. Please write your answers on the exam, not in a blue book. You may use the backs of the pages if necessary. Good luck!

1. True or false (and of course justify):

   (a) If R is an integral domain with quotient field Q then the quotient field of $R[x]$ is isomorphic to $Q[x]$.

   (b) The group $\mathbb{Z}_4 \times \mathbb{Z}_{18}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_{36}$.

2. Let G be a group. Consider the "diagonal"

$$H = \{(x, x) \mid x \in G\} \subset G \times G.$$

H is a subgroup of $G \times G$; you don't have to prove this.

(a) Show that H is a normal subgroup of $G \times G$ if and only if G is abelian.

(b) Assuming G is abelian, show that $(G \times G)/H \simeq G$. (Hint: define a surjective homomorphism $\phi : G \times G \to G$ whose kernel is H.)

3. (a) Find all solutions to the equation $x^2 - 1 = 0$ in $\mathbb{Z}_{35}$.
   (b) Show that if $p > 2$ is prime then either $2^{(p-1)/2} + 1$ or $2^{(p-1)/2} - 1$ is a multiple of p. (Hint: consider the order of 2 in $\mathbb{Z}_p^*$.)

4. (a) Find the quotient and the remainder when $x^3 + 8x^2 + 7x - 1$ is divided by $4x - 1$ in $\mathbb{Z}_{11}[x]$.

   (b) Prove the "remainder theorem": if F is a field, $p \in F[x]$, and $\alpha \in F$, then $p(\alpha)$ is the remainder when p is divided by $x - \alpha$. (Here $p(\alpha)$ denotes the image of p under the evaluation homomorphism $i_\alpha : F[x] \to F$.)

5. True or false (and of course justify):

   (a) The quotient group $(\mathbb{Z} \times \mathbb{Z})/\langle(2,4)\rangle$ is isomorphic to $\mathbb{Z}$.

   (b) There exists a nonzero homomorphism from the group $\mathbb{Z}_{33}$ to the group $\mathbb{Z}_{20}$.