Math 126, K Miller, Spr 2001
Final Exam, May 15 Name Keith Miller

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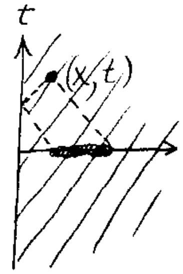


1a Consider the following initial value problem for the 1-D wave eqn on the half line

$$\begin{cases} u_{tt} = u_{xx} & \text{for } x>0, \text{ all } t, \\ u(0,t) = 0 & \text{for all } t, \\ u(x,0) = 0 & \text{for } x>0 \\ u_t(x,0) = \psi(x) & \text{for } x>0. \end{cases}$$

Starting from d'Alembert's formula for the IVP for the whole line, derive the integral formula for $u(x,t)$ here. Explain the shaded interval above.

1b) Among all C^2 functions on the 2-D region D with given Dirichlet bndry values, we want to find that function $u(x,y)$ which minimizes the integral

$$A(u) \equiv \int_D \left(\sqrt{1+u_x^2+u_y^2} + 10u\right) dx\,dy = \text{(surface area of graph) plus (10 times volume under graph).}$$

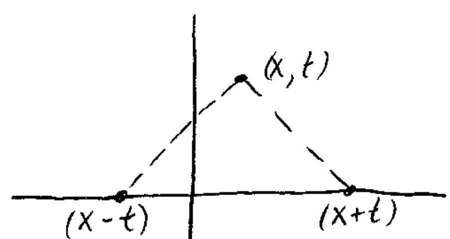
What PDE must u satisfy?

Answer 1a If we extend $\psi(x)$ to the whole line as an odd fn, then solve the IVP on the whole line, the resulting soln will automatically be an odd fn of x for all t and hence will satisfy the bndry condition $u(0,t)=0$ for all t. The d'Alembert soln on the whole line for

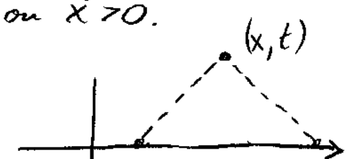
$$\begin{cases} u_{tt} = u_{xx} & \text{for all } x, \text{ all } t \\ u(x,0) = 0 & \text{for all } x \\ u_t(x,0) = \psi_{odd}(x) & \text{for all } x \end{cases}$$

is

$$① \quad u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \psi_{odd}(s)\,ds$$

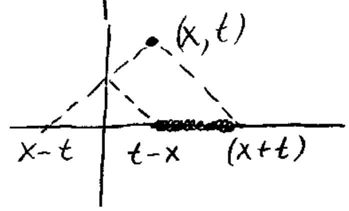


We want this in terms of the original data fn ψ on $x>0$. Thus there are 2 cases:



Case 1 If $x \geq t$ Then ① yields

$$② \quad u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\,ds$$



Case 2 If $x<t$, then the integral of ψ_{odd} in ① from $x-t$ to $t-x$ cancels by antisymmetry so we get

$$③ \quad u(x,t) = \int_{t-x}^{x+t} \psi(s)\,ds$$

Answer 1b This is the general problem of calculus of variations of finding a function u with given bndry values which minimizes the integral functional

$$A(u) \equiv \int_D F(x, u, u_x, u_y)\, dx\, dy.$$

We know that a minimizing fn $u(x,y)$ must satisfy the Euler-Lagrange PDE

$$\frac{\partial F}{\partial u} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right).$$

Here that PDE becomes

$$10 = \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right),$$

since $F = \sqrt{1+u_x^2+u_y^2} + 10u.$

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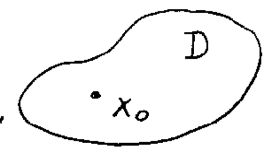
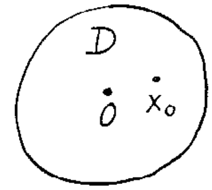
2a The fundamental soln of Laplace's eqn in 3-D is

$$k(x) = \frac{-1}{4\pi}\frac{1}{|x|}.$$

We can smooth-off k to a C^2 function $\tilde{k}$ such that $\tilde{\delta} \equiv \Delta\tilde{k}$ is a "smoothed-off delta fn". What three properties (i), (ii), (iii) does $\tilde{\delta}(x)$ satisfy?



2b Give the defn of the Green's fn $G(x, x_0)$ for a region D in 3-D. (This involves the fundamental solution.)



2c Derive the formula for the Green's fn $G(x, x_0)$ on the unit ball D in 3-D. (State, but don't prove, the identity regarding $|x - x_0|$)

Answer 2a The 3 properties which make $\tilde{\delta} \equiv \Delta\tilde{k}$ an "approx delta fn" are

(i) $\int_{R^3} \tilde{\delta}(x)\,dx = 1$ (ii) $\tilde{\delta}(x) \geq 0$ (iii) support of $\tilde{\delta}$ is in the ϵ-ball about the origin. Incidentally (i) holds for any way to smooth-off k to a C^2 fn $\tilde{k}$, because in a ball B_R with $R > \epsilon$ we have

$$\int_{B_R} \Delta\tilde{k} \overset{\text{"Green's Oth"}}{=} \int_{\partial B_R} \frac{\partial \tilde{k}}{\partial n}\,dS = \int_{\partial B_R} \frac{\partial k}{\partial n}\,dS = 1.$$

Note 1 Because of (i)–(iii), if x_0 is in D and u is contin at x_0, then

$$\int_D u(x)\,\tilde{\delta}(x - x_0)\,dx \to u(x_0) \text{ as } \epsilon \to 0.$$

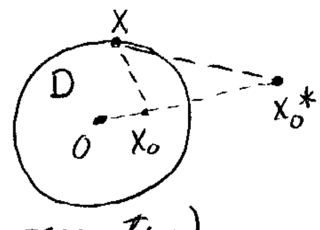
Answer 2b

Defn Let x_0 in D be fixed. $G(x, x_0)$ is that function of the form

① $G(x, x_0) = k(x - x_0) + W(x, x_0)$, where

② W is a $C^2(\bar{D})$ harmonic fn of x in D, and

③ $G(x, x_0) = 0$ for x on ∂D (ie W on ∂D equals $-k(x - x_0)$)



Answer 2c The "reflection pt" x_0^* is that point on the ray thru x_0 such that $|x_0|\,|x_0^*| = 1$.
Thus $x_0^* = \frac{1}{|x_0|^2}\, x_0$

- Identity ④ $|x_0|\,|x - x_0^*| = |x_0|$ for all x on ∂D (this is easily verified by 2-D high-school analytic geometry)
- Thus our Green's fn is

$$G(x, x_0) = \underbrace{-\frac{1}{4\pi}\left(\frac{1}{|x - x_0|}\right)}_{k(x - x_0)} + \underbrace{\frac{1}{4\pi}\left(\frac{1}{|x_0|\,|x - x_0^*|}\right)}_{W(x, x_0) \equiv \frac{1}{|x_0|} k(x - x_0^*)}$$

Because W has its singularity outside D, we satisfy ②.
Because $W = -k(x - x_0)$ for x on ∂D, we satisfy ③.

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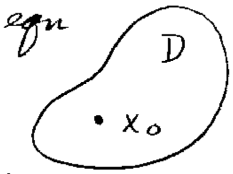

3 Suppose u is the soln of the Dirichlet problem for Poisson's eqn

$$(1) \begin{cases} \Delta u = \varphi(x) \text{ in } D, \\ u = f(x) \text{ on } \partial D. \end{cases}$$

State and prove an integral formula for $u(x_0)$. (Involving the Green's fn $G(x,x_0)$.) Your proof should involve the "smoothed-off fundamental soln" $\tilde{k}(x-x_0)$ from problem 2a.

Answer Let $\tilde{G}(x,x_0) \equiv \tilde{k}(x-x_0) + W(x,x_0)$ be the "smoothed-off Green's fn". It satisfies

(1) $\Delta \tilde{G}(x,x_0) = \Delta \tilde{k}(x-x_0) + 0 = \tilde{\delta}(x-x_0)$ in D

(2) $\tilde{G}(x,x_0) = 0$ for x on ∂D.

(3) $\tilde{G}$ is a $C^2(\bar{D})$ fn of x

Plugging $v(x) \equiv \tilde{G}(x,x_0)$ into Greens 2nd identity we have

$$(4) \int_D \underbrace{\Delta u}_{\varphi(x)} \tilde{G} - u \underbrace{\Delta \tilde{G}}_{\tilde{\delta}(x-x_0)} \, dx = \int_{\partial D} \left(\frac{\partial u}{\partial n} \underbrace{\tilde{G}}_{=G=0 \text{ on } \partial D} - \underbrace{u}_{=f(x)} \underbrace{\frac{\partial \tilde{G}}{\partial n}}_{=\frac{\partial G}{\partial n} \text{ on } \partial D} \right) dS$$

Thus.

$$(5) \int_D \varphi(x) \tilde{G}(x,x_0) - u(x) \tilde{\delta}(x-x_0) \, dx = -\int_{\partial D} f(x) \frac{\partial G}{\partial n}(x,x_0) \, dS$$

Note 2 The $O(\frac{1}{r})$ singularity of $G(x,x_0)$ is sufficiently weak that $\int_D \varphi(x) \tilde{G}(x,x_0) \, dx \to$ the improper integral $\int_D \varphi(x) G(x,x_0) \, dx$ as $\epsilon \to 0$.

Using Note 2 and the Note 1 of Answer 2a, we take the limit as $\epsilon \to 0$ in (5) to get the desired formula

$$u(x_0) = \int_{x \in D} \varphi(x) G(x,x_0) \, dx + \int_{x \in \partial D} f(x) \frac{\partial G}{\partial n}(x,x_0) \, dS$$

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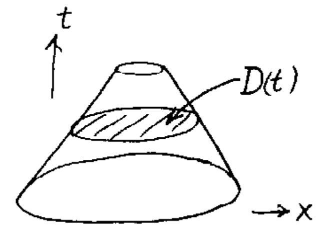
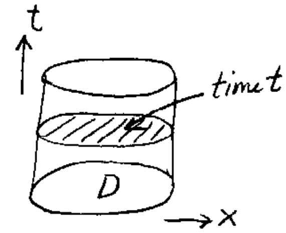


4 Let $u(x,t)$ be a C^2 soln of the wave eqn $u_{tt} = \Delta u$ on the $(n+1)$-dimensional pyramidal region shown, where the bndry $\partial D(t)$ of the cross section $D(t)$ is receding with a normal velocity $v(x,t)$ which is $\geq$ the wave speed 1. Show that the "energy" $E(t)$ of the soln in $D(t)$ satisfies $E'(t) \leq 0$.
(You may apply, without explanation, my formula for the t-derivative of an integral on a changing region $D(t)$.)



Alternative for 50% credit, if you can't do the above: Let $u(x,t)$ be a C^2 soln of the wave eqn in the cylindrical region shown (ie D is fixed) and let $u(x,t) \equiv 0$ on the bndry of D. Show that the "energy" $E(t)$ of the soln remains constant.

Answer

① $$E(t) \equiv \int_{D(t)} \underbrace{\tfrac{1}{2}(u_t^2 + \nabla u \cdot \nabla u)}_{\equiv \rho(x,t) \text{ - the "energy density", } \geq 0.}\, dx$$

My formula for the t-derivative on this receding region is

② $$E'(t) = \int_{D(t)} \frac{\partial}{\partial t}(\rho(x,t))\,dx - \int_{\partial D(t)} \rho(x,t)\, v(x,t)\, dS$$

Here this becomes

$$E'(t) = \int_{D(t)} (u_t \underbrace{u_{tt}}_{= \Delta u \text{ by the PDE}} + \nabla u \cdot (\nabla u)_t)\,dx - \int_{\partial D(t)} \underbrace{\rho(x,t)}_{\geq 0}\ \underbrace{v(x,t)}_{\geq 1}\, dS$$

Thus

$$E'(t) \leq \int_{D(t)} \underbrace{u_t\, \Delta u}_{\text{apply Greens 1st here}} + \nabla u \cdot (\nabla u)_t\, dx - \int_{\partial D(t)} \tfrac{1}{2}(u_t^2 + |\nabla u|^2)\, dS =$$

$$\overset{\text{Green's}}{\underset{1st}{=}} \int_{D(t)} -\nabla(u_t)\cdot \nabla u + \nabla u \cdot \underbrace{(\nabla u)_t}_{= \nabla u_t \text{ since } u \text{ is } C^2} dx + \int_{\partial D(t)} u_t \frac{\partial u}{\partial n} - \int \tfrac{1}{2}(u_t^2 + \underbrace{|\nabla u|^2}_{\geq \left(\frac{\partial u}{\partial n}\right)^2})\, dS$$

$$\leq 0 - \int \tfrac{1}{2}\left(u_t^2 - 2u_t \frac{\partial u}{\partial n} + \left(\frac{\partial u}{\partial n}\right)^2\right) dS = -\int_{\partial D(t)} \tfrac{1}{2}\left(u_t - \frac{\partial u}{\partial n}\right)^2 dS \leq 0.$$

qed

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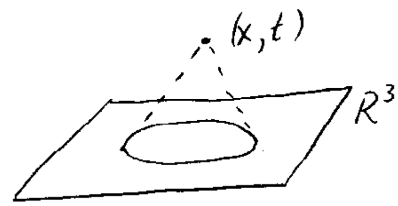

5a Kirchoff's formula for the soln of

$$\text{(A)}\begin{cases} u_{tt} = c^2 \Delta u & \text{on all } R^4 \\ u(x,0) = 0 & \\ u_t(x,0) = \psi(x) & \end{cases} \text{ at } t=0$$

is

(1) $u(x,t) =$

5b Use (1) to derive that the soln of

$$\text{(B)}\begin{cases} u_{tt} = c^2 \Delta u & \text{on all } R^4 \\ u(x,0) = \varphi(x) & \\ u_t(x,0) = 0 & \end{cases} \text{ at } t=0$$

is given by

(2) $u(x,t) =$

Answer 5a (1) $u(x,t) = t \cdot [MV \text{ of } \psi \text{ on } S]$ where S is the spherical shell of radius ct about center x in R^3.

Answer 5b Let W be the soln of

$$\text{(C)}\begin{cases} W_{tt} = c^2 \Delta W & \text{on all } R^4 \\ W(x,0) = 0 & \\ W_t(x,0) = \varphi(x) & \end{cases}$$

Then I claim that (if φ is C^3, so that W is also C^3) the t-derivative W_t solves (B). This is because we can differentiate the PDE in (C) and bring the $(\ )_t$ inside the Δ (all because W is C^3) to get

$$\begin{cases} \bullet \quad (W_{tt})_t = (c^2 \Delta W)_t \quad \text{on all } R^4 \\ \qquad\ \ \| \qquad\qquad \| \text{ since } W \text{ is } C^3 \\ \quad\ (W_t)_{tt} \qquad c^2 \Delta(W_t) \\ \bullet \quad (W_t)(x,0) = \varphi(x) \\ \bullet \quad (W_t)_t(x,0) = W_{tt} \overset{\text{by PDE (C)}}{=} \underbrace{c^2 \Delta W}_{\substack{\text{but } W \equiv 0 \text{ at } t=0 \text{ by (C)},\\ \text{so this is } \equiv 0}} = 0 \end{cases}$$

Thus

(2) $u(x,t) = W_t(x,t) \overset{\text{by (1)}}{=} \dfrac{\partial}{\partial t}\left\{ t \cdot [MV \text{ of } \varphi \text{ on } S] \right\}$

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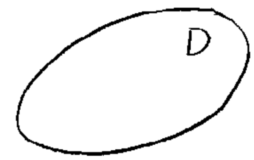
6 Consider the Dirichlet problem for Poisson's eqn

$$(1)\quad \begin{cases} -\Delta u = f(x) \text{ in } D, \\ u = 0 \quad \text{on } \partial D. \end{cases}$$



6a What is the function space $pw\,C^1_z(\bar{D})$?
Give the defn of "u is a weak soln of (1)".

6b Let M_h be a finite dimensional subspace of $pw\,C^1_z(\bar{D})$. Define the Galerkin or Finite Element approximate soln U.

6c State and prove the "best approximation property in energy norm" for the FE soln U. (But first define the "energy inner product and norm $((\ ,\ ))$ and $|||\ \ |||$.")

Answer 6a $pw\,C^1_z(\bar{D})$ is the space of continuous fns on $\bar{D}$ which are also piecewise C^1 on $(\bar{D})$.

Now if u is a classical $C^2(\bar{D})$ soln of (1), we multiply the PDE (1) by any test fn φ in $pw\,C^1_z(\bar{D})$, then use Greens 1st and the zero bndry values to get that u satisfies

$$(2)\quad \underbrace{\int_D \nabla u \cdot \nabla \varphi \, dx}_{\text{defn } \equiv a(u,\varphi) - \text{the bilinear form for } -\Delta} = \underbrace{\int f \varphi \, dx}_{\equiv (f,\varphi)}$$

Defn For any two fns u, v in $pw\,C^1_z(\bar{D})$ we define the "energy" inner product and norm

$$((u,v)) \equiv a(u,v), \quad |||u||| \equiv ((u,u))^{\frac{1}{2}}$$

Defn u is a weak soln of (1) if

$$(3)\quad \begin{cases} u \in pw\,C^1_z(\bar{D}), \\ a(u,\varphi) = (f,\varphi) \quad \text{for all test fns } \varphi \text{ in } pw\,C^1_z(\bar{D}) \end{cases}$$

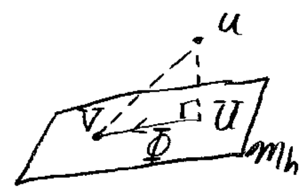
Answer 6b Defn The FE soln U is the unique soln of

$$(4)\quad \begin{cases} U \in M_h \\ a(U,\Phi) = (f,\Phi) \quad \text{for all test fns } \Phi \text{ in } M_h. \end{cases}$$

Answer 6c

- Since $((\,,\,))$ is an inner product on $pw\, C^1_z(\bar{D})$ we have the usual Cauchy-Schwarz inequality $((v, w)) \le |||v|||\ |||w|||$
- Since we can use test fns Φ in ③ also, subtracting ④ from ③ we get

$$⑤\ ((u-U, \Phi)) \equiv a(u-U, \Phi) = 0 \text{ for all } \Phi \text{ in } M_h.$$



Thm (Best approx property) For any elt V in M_h we have $|||u-U||| \le |||u-V|||$

Proof $\Phi \equiv V - U$ is in M_h. Hence.

$$\|u-U\|^2 = ((u-U, \underbrace{u-U}_{u-V+\Phi})) = ((u-U, u-V)) + \underbrace{((u-U, \Phi))}_{=0 \text{ by } ⑤}$$

$$\underset{\substack{\text{Schwarz}\\ \text{Inequality}}}{\overset{\text{Cauchy}}{\le}} |||u-U|||\ |||u-v|||.$$

Better Proof Use Pythagoras and the orthogonality of $u-U$ and ϕ in the energy inner product, as shown in the diagram.

$$|||u-V|||^2 = |||(u-U) + \Phi|||^2 \equiv ((u-U) + \Phi, (u-U) + \Phi))$$

$$= |||u-u|||^2 + 2\underbrace{((u-U, \Phi))}_{=0 \text{ by } ⑤} + \underbrace{|||\Phi|||^2}_{>0 \text{ if } \Phi \neq 0}$$

$$> |||u-U|||^2.$$

qed

END