MATH 114

PROFESSOR KENNETH A. RIBET

Final Examination

May 21, 2004

12:30–3:30 PM

70 Evans

Name:

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. If you invoke a theorem that we proved in class, state the theorem clearly and explain carefully how you are applying it. You may use the equation $999 = 27 \cdot 37$ without justifying it. Your paper is your ambassador when it is graded.

Problem	Your score	Total points
1		5 points
2		20 points
3		5 points
4		5 points
5		5 points
6		5 points
Total:		45 points

1. Let $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ be a monic polynomial with integer coefficients. Suppose that $f(t)$ is the product $g(t)h(t)$ where g and h are monic polynomials with *rational* coefficients. Show that g and h have integral coefficients.

2. In each of the following five situations, find the Galois group $G = \mathrm{Gal}(\Sigma/K)$, where Σ is the splitting field of $f(t)$ over K. If you are able to calculate only $[\Sigma : K]$, but not G, please provide this degree. The symbol "p" denotes a prime number. (Note that this problem spans three pages.)

a. $K = \mathbb{Q}$, $f(t) = t^{16} + 1$.

b. $K = \mathbb{Q}(e^{2\pi i/5})$, $f(t) = t^5 - 100$.

c. $K = \mathbb{F}_2$, $f(t) = t^3 + t + 1$.

d. $K = \mathbb{F}_p$, $f(t) = (t^2 - 1)(t^2 - 2) \cdots (t^2 - (p-1))$.

e. $K = \mathbb{Q}$, $f(t) = t^5 - 10t + 5$. (You might want to show that this polynomial has exactly 3 real roots.)

3. Let $\zeta = e^{2\pi i/37}$, $\alpha = \zeta + \zeta^{10} + \zeta^{26}$. Use Galois theory to calculate the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

4. What does it mean to say that a finite extension of fields is *separable*? Construct a finite extension that is not separable, and explain in detail why the construction yields what is required.

5. Suppose that G is a finite group. For each $g \in G$, let $S_g = \{\, xgx^{-1} \mid x \in G \,\}$. Show that the number of elements of S_g is a divisor of the order of G.

6. Suppose that $L : K$ is a Galois extension and that E is a subfield of L that contains K. Assume that L is the smallest subfield of L that contains E and is Galois over K (i.e., that no proper subfield of L has this property). Show that no proper subfield of L contains all the fields $\sigma(E)$ as σ runs over $\mathrm{Gal}(L : K)$.