Math 110 Section 2, Spring 2001 FINAL EXAM B. Curtin

Problems will be graded for correctness/completeness, so be sure to include explanations which justify each step in your arguments and computations. For example, check that the hypotheses of any theorem that you use are satisfied. Do not interpret the problems in such a way that they become trivial–if in doubt, ask. Problems will also be graded for for how well they are written. Ideally, you should use complete sentences to explain your ideas. Of course equations can be included in such a proof.

Should it become necessary to leave the room during this exam (eg. fire alarm), this exam and all your work is to remain in the room, face down on your desk.

1. Let A be a 2×2 real matrix with eigenvalues 1 and 0 corresponding to respective eigenvectors $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Show that A is symmetric.
2. Let T be a linear operator on a finite dimensional complex inner product space V. Let $x \in V$. Show that if T is self-adjoint, then $T^2(x) = 0$ implies that $T(x) = 0$.
3. Let V and W denote vector spaces over a field F. Let T and U denote nonzero linear transformations from V into W such that $R(T) \cap R(U) = \{0\}$. Prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.
4. Prove that the eigenvectors of a complex normal matrix which correspond to distinct eigenvalues are orthogonal.
5. Let A be a matrix over a field F. Prove that if λ is an eigenvalue of A, then λ is an eigenvalue of A^t with the same algebraic and geometric multplicities.
6. Let A be an $n \times n$ matrix over a field F. Prove that
$$\dim(\operatorname{span}\{I,\ A,\ A^2,\ \ldots\}) \leq n.$$
7. Let V denote an inner product space. Fix $y, z \in V$ and define $T : V \to V$ by $T(x) = \langle x, y \rangle z$. Show that T is linear and that T^* exists. Give an expression for $T^*(x)$ involving x, y, and z.
8. Let A and B be $n \times n$ matrices over a field F. Prove that if $AB = I$ then $BA = I$. (Here you are to use the definition of inverse, just as you did when you did this problem in homework).
9. Let
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$
Observe that A is a real symmetric matrix so it is orthogonally equivalent to a diagonal matrix (you don't have to do anything about this comment). Find a diagonal matrix D and an orthogonal matrix Q such that $Q^{-1}AQ = D$.