Math 53: Multivariable Calculus Worksheets

7^{th} Edition

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Preface

This booklet contains the worksheets for Math 53, U.C. Berkeley's multivariable calculus course.

The introduction of each worksheet very briefly summarizes the main ideas but is not intended as a substitute for the textbook or lectures. The questions emphasize qualitative issues and the problems are more computationally intensive. The additional problems are more challenging and sometimes deal with technical details or tangential concepts.

Typically more problems were provided on each worksheet than can be completed during a discussion period. This was not a scheme to frustrate the student; rather, we aimed to provide a variety of problems that can reflect different topics which professors and GSIs may choose to emphasize.

The first edition of this booklet was written by Greg Marks and used for the Spring 1997 semester of Math 53W.

The second edition was prepared by Ben Davis and Tom Insel and used for the Fall 1997 semester, drawing on suggestions and experiences from the first semester. The authors of the second edition thank Concetta Gomez and Professors Ole Hald and Alan Weinstein for their many comments, criticisms, and suggestions.

The third edition was prepared during the Fall of 1997 by Tom Insel and Zeph Grunschlag. We would like to thank Scott Annin, Don Barkauskas, and Arturo Magidin for their helpful suggestions. The Fall 2000 edition has been revised by Michael Wu.

Tom Insel coordinated this edition in consultation with William Stein.

Michael Hutchings made tiny changes in 2012 for the seventh edition.

In 1997, the engineering applications were written by Reese Jones, Bob Pratt, and Professors George Johnson and Alan Weinstein, with input from Tom Insel and Dave Jones. In 1998, applications authors were Michael Au, Aaron Hershman, Tom Insel, George Johnson, Cathy Kessel, Jason Lee, William Stein, and Alan Weinstein.

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1. Curves Defined by Parametric Equations

As we know, some curves in the plane are graphs of functions, but not all curves can be so expressed. Parametric equations allow us to describe a wider class of curves. A parametrized curve is given by two equations, $x = f(t)$, $y = g(t)$. The curve consists of all the points (x, y) that can be obtained by plugging values of t from a particular domain into both of the equations $x = f(t)$, $y = g(t)$. We may think of the parametric equations as describing the motion of a particle; $f(t)$ and $g(t)$ tell us the x- and y-coordinates of the particle at time t. We can also parametrize curves in $\mathbf{R}^3$ with three parametric equations: $x = f(t)$, $y = g(t)$, and $z = h(t)$. For example, the orbit of a planet around the sun could be given in this way.

Questions

1. (a) Check that the graph of the function $y = x^2$ is the same as the parametrized curve $x = t$, $y = t^2$.
   (b) Using (a) as a model, write parametric equations for the graph of $y = f(x)$ where $f(x)$ is any function.
2. Consider the circle $C = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$.
   (a) Is C the graph of some function? If so, which function? If not, why not?
   (b) Find a parametrization for C. (**Hint:** $\cos^2\theta + \sin^2\theta = 1$.)
3. Consider the parametric equations $x = 3t$, $y = t$, and $x = 6t$, $y = 2t$.
   (a) What curves do the two sets of equations describe?
   (b) Compare and contrast the motions for the two sets of parametric equations by interpreting each set as describing the motion of a particle.
   (c) Suppose that a curve is parametrized by $x = f(t)$, $y = g(t)$. Explain why $x = f(2t)$, $y = g(2t)$ parametrize the same curve.
   (d) Show that there are an infinite number of different parametrizations for the same curve.

Problems

1. Consider the curve parametrized by $x(\theta) = a\cos\theta$, $y(\theta) = b\sin\theta$.
   (a) Plot some points and sketch the curve when $a = 1$ and $b = 1$, when $a = 2$ and $b = 1$, and when $a = 1$ and $b = 2$.
   (b) Eliminate the parameter θ to obtain a single equation in x, y, and the constants a and b. What curve does this equation describe? (**Hint:** Eliminate θ using the identity $\cos^2\theta + \sin^2\theta = 1$.)

2. Consider the parametric equations $x = 2\cos t - \sin t$, $y = 2\cos t + \sin t$.

   (a) Eliminate the parameter t by considering $x + y$ and $x - y$.

   (b) Your result from part (a) should be quadratic in x and y, and you can put it in a more familiar form by substitution $x = u + v$ and $y = u - v$. Which sort of conic section does the equation in u and v describe?

3. Let C be the curve $x = t + 1/t$, $y = t - 1/t$.

   (a) Show that C is a hyperbola. (**Hint:** Consider $(x + y)(x - y)$.)

   (b) Which range of values of t gives the left branch of the hyperbola? The right branch?

   (c) Let D be the curve $x = t^2 + 1/t^2$, $y = t^2 - 1/t^2$. How does D differ from C? Explain the difference in terms of the parametrizations.

Additional Problems

1. A *helix* is a curve in the shape of a corkscrew. Parametrize a helix in $\mathbf{R}^3$ which goes through the points $(0, 0, 1)$ and $(1, 0, 1)$.

2. Tangents, Areas, Arc Lengths, and Surface Areas

As we saw in the previous section, we can use parametric equations to describe curves that aren't graphs of the form $y = f(x)$. In Math 1A we learned how to calculate the slope of a graph at a point and how to evaluate the area underneath a graph. In Math 1B, we encountered the problems of calculating the arc length of a graph and the area of a surface of revolution defined by a graph. Here, we revisit these problems in the more general framework of parametrized curves.

Here are *some* of the new formulas:

- The slope of a parametrized curve (when $\frac{dx}{dt} \neq 0$):

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \tag{1}$$

- The arc length of a curve parametrized for $\alpha \leq t \leq \beta$:

$$\int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{2}$$

- The area obtained by rotating a curve parameterized for $\alpha \leq t \leq \beta$ around the x-axis.

$$\int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{3}$$

Questions

1. A Geometric Proof of Equation 1. Let C be the curve given by the parametric equations $x = f(t)$, $y = g(t)$ and let $(f(t_0), g(t_0))$ be a point on the curve. Let $m(t)$ be the slope of the secant line connecting $(f(t_0), g(t_0))$ to $(f(t), g(t))$ as in Figure 1.

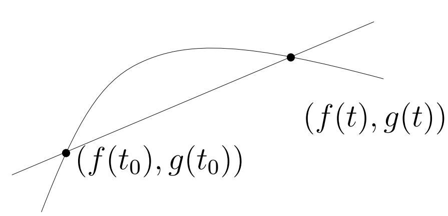


Figure 1: A secant line

(a) Write a formula for $m(t)$ in terms of $f(t)$ and $g(t)$.

(b) Use l'Hospital's rule to evaluate $\lim_{t \to t_0} m(t)$ and explain how this limit gives the slope of the tangent line.

2. In Question 3(d) on Worksheet 1, you found an infinite number of parametrizations for a single curve. Verify that Formula 1 yields the same tangent slope to the curve at a point, no matter which of the parametrizations is used.

3. Show that Formula 2 recovers the usual formula

$$\text{arc length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dt$$

   in the special case when the curve is the graph of a function $y = f(x)$, $a \leq x \leq b$.

Problems

1. Let C be the curve $x = 2\cos t$, $y = \sin t$.
   (a) What kind of curve is this?
   (b) Find the slope of the tangent line to the curve when $t = 0$, $t = \pi/4$, and $t = \pi/2$.
   (c) Find the area of the region enclosed by C. (**Hint:** $\sin^2 t = (1 - \cos 2t)/2$.)

2. Compute the arc length of the curve parametrized by $x = \cos(e^t)$, $y = \sin(e^t)$, $0 \leq t \leq 1$. (**Hint:** Reparametrize.)

3. Consider the circle parametrized by $x = 2 + \cos t$, $y = \sin t$, $0 \leq\leq 2\pi$.
   (a) What are the center and radius of this circle.
   (b) Describe the surface obtained by rotating the circle about the x-axis? About the y-axis.
   (c) Calculate the area of the surface obtained by rotating the circle around the x-axis. Why should you restrict the parametrization to $0 \leq t \leq \pi$ when integrating?
   (d) Calculate the area of the surface obtained by rotating the circle about the y-axis.

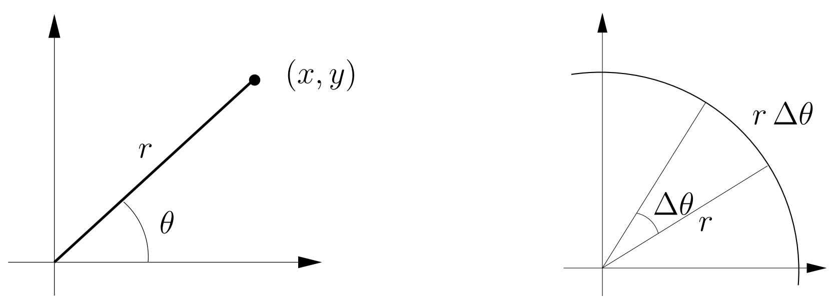
Additional Problem

1. Consider the surface obtained by rotating the parametrized curve $x = e^t + e^{-t}$, $y = e^t - e^{-t}$, $0 \leq t \leq 1$ about the x-axis.
   (a) Find the area of this surface by plugging x and y into Formula 3 and integrating by substitution.
   (b) Find the area of this surface by reparametrizing the curve before you plug into Formula 3.

3. Polar Coordinates

Polar coordinates are an alternative to Cartesian coordinates for describing position in $\mathbf{R}^2$. To specify a point in the plane we give its distance from the origin (r) and its angle measured counterclockwise from the x-axis (θ). Polar coordinates are usually used when the region of interest has circular symmetry. Curves in polar coordinates are often given in the form $r = f(\theta)$; if we wish to find tangent lines, areas or other information associated with a curve specified in polar coordinates, it is often helpful to convert to Cartesian coordinates and proceed as in Sections 9.2 and 9.3.

The area of a region in polar coordinates can be found by adding up areas of "infinitesimal circular sectors" as in Figure 2(b). The area inside the region bounded by the rays $\theta = a$ and $\theta = b$ and the curve $r = f(\theta)$ is $\int_a^b \frac{1}{2} f^2(\theta)\, d\theta$.



(a) A point (b) "Infinitesimal sector"

Figure 2:

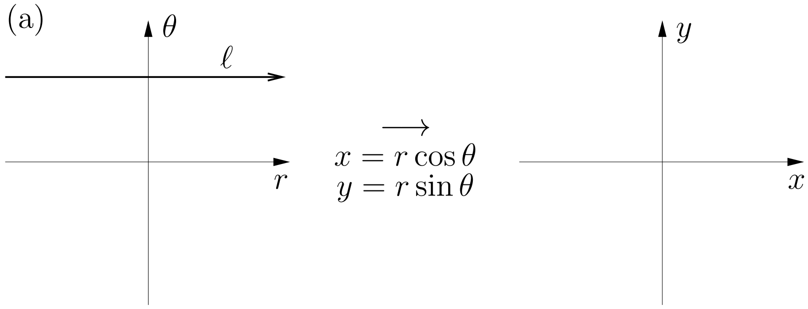
Questions

1. (a) Find the x- and y-coordinates in terms of r and θ for the point in Figure 2(a).

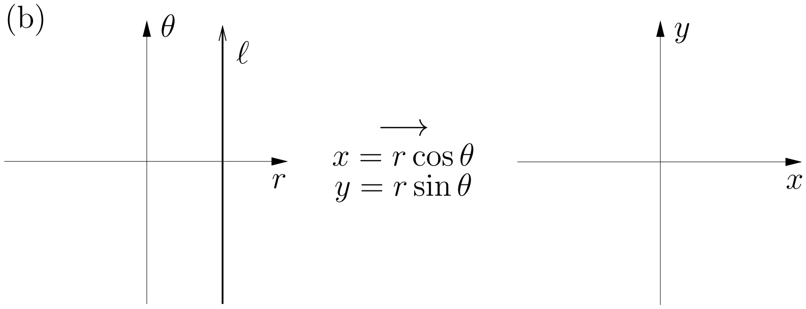
   (b) Find the r- and θ-coordinates in terms of x and y for the point in Figure 2(a).

2. The polar coordinate map $x = r\cos\theta$, $y = r\sin\theta$ takes a line in the $r\theta$-plane to a curve in the xy-plane. For each line ℓ, draw the corresponding curve in the xy-plane.

(a)



(b)



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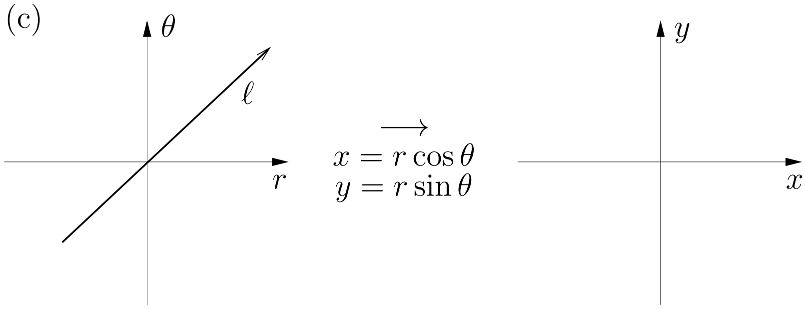


Problems

1. Sketch the curve given by $r = 2\sin\theta$ and give its equation in Cartesian coordinates. What curve is it?

2. Write an equation in polar coordinates for the circle of radius $\sqrt{2}$ centered at $(x, y) = (1, 1)$.

3. Consider the curve given by the polar equation $r = 3 + \cos 4\theta$.

   (a) Sketch this curve.

   (b) Find the slope of this curve at $\theta = \pi/4$.

   (c) At which points does $\frac{dr}{d\theta} = 0$? Remember that this is *not* the slope. What is the geometric meaning of $\frac{dr}{d\theta} = 0$?

4. (a) Does the spiral $r = 1/\theta$, $\pi/2 \le \theta < \infty$ have finite length?

(b) Does the spiral $r = e^{-\theta}$, $0 \leq \theta < \infty$ have finite length?

5. Sketch the *lemniscate* $r^2 = a^2 \cos(2\theta)$ where a is a positive constant and calculate the area it encloses.

4. Vectors, Dot Products, Cross Products, Lines and Planes

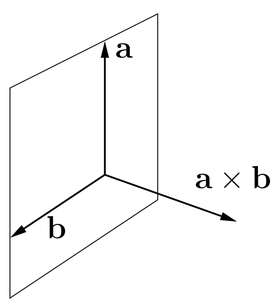
In engineering and the physical sciences, a vector is any quantity possessing both magnitude and direction. Force, displacement, velocity, and acceleration are all examples of vectors.

One way of multiplying two vectors together is the *dot product*. If $\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n\rangle$ then $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$. Note that the result of the dot product is a number, not a vector. The dot product gives an easy way of computing the angle between two vectors: the relationship is given by the formula $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. In particular, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

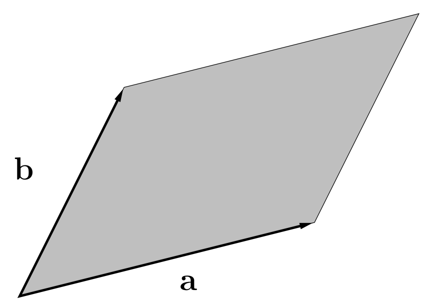
Another way of multiplying vectors in $\mathbf{R}^3$ is the *cross product*. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Note that the result of the cross product is a vector, not a number. The cross product reflects several interesting geometric quantities. First, it gives the angle between the vectors by the formula $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$. Second, the vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. Third, $|\mathbf{a}\times\mathbf{b}|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.



(a) Vector perpendicular to a plane



(b) Parallelogram spanned by **a** and **b**

Figure 3: The cross product

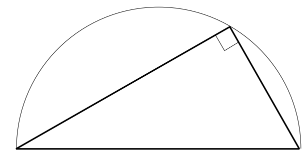
Questions

1. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^3$. Can $\mathbf{u}\times\mathbf{v}$ be a non-zero scalar multiple of $\mathbf{u}$?

2. Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbf{R}^3$. Show that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$. What is the name of this famous theorem?

3. Find a vector perpendicular to $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ in $\mathbf{R}^3$. Draw a picture to illustrate that there are many correct answers.

4. (a) Suppose that one side of a triangle forms the diameter of a circle and the vertex opposite this side lies on a circle. Use the dot product to prove that this is a right triangle.



   (b) Now, do the same in $\mathbf{R}^3$.
   (**Hint:** Let the center of the circle be the origin.)

Problems

1. Here we find parametric equations for the line in $\mathbf{R}^3$ passing through the points $\mathbf{a} = (1, 0, 1)$ and $\mathbf{b} = (2, 1, -1)$.
   (a) Find a vector $\mathbf{u}$ pointing in the same direction as the line.
   (b) Let $\mathbf{c}$ be any point on the line. Explain why $\mathbf{c} + t\mathbf{u}$ gives parametric equations for the line. Write down these equations
   (c) Can you get more than one parametrization of the line from these methods?
2. Consider the plane $x + y - z = 4$.
   (a) Find any point in the plane and call it $\mathbf{a}$. Let $\mathbf{x} = (x, y, z)$ and show that $(\mathbf{x} - \mathbf{a}) \cdot (1, 1, -1) = 0$ is the equation of the plane.
   (b) Explain why $\mathbf{i} + \mathbf{j} - \mathbf{k}$ is a normal vector to the plane.
   (c) Show that if $ax + by + cz = d$ is the equation of a plane where a, b, c, and d are constants, then $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a normal vector.
3. Here we find the equation of a plane containing the points $\mathbf{a} = (0, 0, 1)$, $\mathbf{b} = (0, 1, 2)$ and $\mathbf{c} = (1, 2, 3)$.
   (a) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors connecting $\mathbf{a}$ to $\mathbf{b}$ and $\mathbf{a}$ to $\mathbf{c}$. Compute $\mathbf{u}$ and $\mathbf{v}$.
   (b) Find a vector perpendicular to the plane.
   (c) Use the normal vector to find the equation of the plane.
   (**Hint:** First write the equation in the form given in Problem 2(a).)

Additional Problems

1. Suppose that you are looking to the side as you walk on a windless, rainy day. Now you stop walking.[1]
   (a) How does the apparent direction of the falling rain change?
   (b) Explain this observation in terms of vectors.
   (c) Suppose you know your walking speed. How could you determine the speed at which the rain is falling?

[1] From *Basic Multivariable Calculus* by Marsden, Tromba, and Weinstein.

5. Quadric Surfaces

The quadric surfaces in $\mathbf{R}^3$ are analogous to the conic sections in $\mathbf{R}^2$. Aside from *cylinders*, which are formed by "dragging" a conic section along a line in $\mathbf{R}^3$, there are only six quadric surfaces: the *ellipsoid*, the *hyperboloid of one sheet*, the *hyperboloid of two sheets*, the *elliptic cone*, the *elliptic paraboloid*, and the *hyperbolic paraboloid*. We study quadric surfaces now because they will provide a nice class of examples for calculus.

Questions

1. For each of the six types of quadric surfaces listed above, which of the following is true?

   ① Every surface of this type can be formed by rotating some curve about an axis.
   ② Some surfaces of this type can be so formed and some cannot.
   ③ No surface of this type can be so formed.

2. Which type of quadric surface is given by the equation $x^2 + y^2 = 1$?
   (**Hint:** It is not a circle.)

Problems

1. Which type of quadric surface is given by each of these equations?

   (a) $x^2 + 2y^2 + 3z^2 + x + 2y + 3z = 0$
   (b) $x^2 + 2y^2 - 3z^2 + x + 2y + 3z = 0$
   (c) $x^2 - 2y^2 - 3z^2 + x + 2y + 3z = 0$

2. Which type of quadric surface is $z = xy$?

3. Show that the quadric surface $xy + xz + yz = 0$ is a cone.
   (**Hint:** Begin by showing that if (a, b, c) is a point on this surface, then (ka, kb, kc) is another point on the surface for every number k.)

6. Vector Functions and Space Curves

A *vector-valued function* is any function $f\colon \mathbf{R} \to \mathbf{R}^n$, that is, any functions which takes a number and outputs a vector. For example, given a parametrized curve $x = f(t)$, $y = g(t)$, $z = h(t)$, we can construct the *position vector* $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. Geometrically, $\mathbf{r}(t)$ is an arrow from the origin to the point $(f(t), g(t), h(t))$ as shown in Figure 4(a).

The derivative of a vector-valued function is defined in a familiar way:

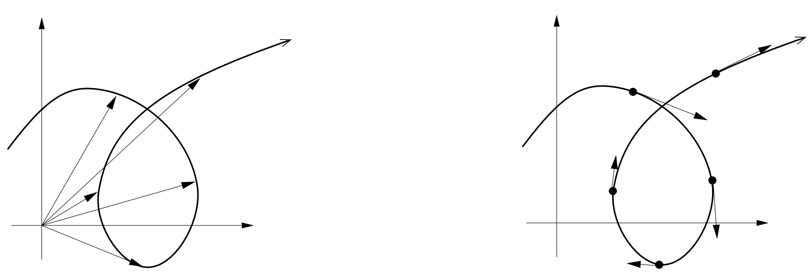
$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}.$$

As a practical matter, the derivative of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ can be shown to be $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$, that is, we take the derivative *component-wise*. The derivative of the position vector is called the *velocity vector*. If we think of the position vector as giving the position of a particle, the velocity vector at point of the curve is an arrow pointing in the direction of motion, whose length is equal to the speed of the particle (see Figure 4(b)).

Since the distance traveled by a particle is equal to the integral of its speed with respect to time, we have the following formula for the arc length of a parametrized curve:

$$\text{arc length} = \int_a^b |r'(t)|\, dt$$

where a and b are the start and end times.



(a) Position vectors (b) Velocity vectors

Figure 4:

Questions

1. Show that if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is the position vector of a parametrized curve in $\mathbf{R}^2$ then the preceding definition of arc length as the integral of speed agrees with Formula 2 on Worksheet 2.

2. Since the velocity vector of a curve in the plane is tangent to the curve, it should be possible to use it to get the slope of the tangent line. Do so for the parametrized curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$.

3. Describe what it means for:
   (a) $\mathbf{r}(t)$ to be constant.
   (b) $\mathbf{r}'(t)$ to be constant.
   (c) $|\mathbf{r}(t)|$ to be constant.
   (d) $|\mathbf{r}'(t)|$ to be constant.

Problems

1. Let $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0\mathbf{k}$.
   (a) Is $\mathbf{r}(t)$ perpendicular to $\mathbf{r}'(t)$ for every t?
   Is $\mathbf{r}'(t)$ perpendicular to $\mathbf{r}''(t)$ for every t?
   (b) If $\mathbf{r}$ were another function, would the two answers to (a) remain the same? If true, show why. If false, give a counter example.
2. Suppose that a particle's position vector is given by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. Find its position, velocity, speed, and acceleration when $t = 10$.
3. Find the tangent line to the curve $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^4\mathbf{k}$ at the point $(1, 1, 1)$.

Additional Problems

1. An object starting at the origin and moving under the influence of gravity, but no other forces, follows the trajectory $x(t) = at$, $y(t) = bt$, $z(t) = ct - \frac{1}{2}gt^2$, where $g \approx 9.8\text{m/s}^2$ is the gravitational acceleration constant and a, b, and c are constants depending on the particular motion.
   (a) Find the velocity vector as a function of time.
   (b) What is the meaning of the constants a, b, and c?
   (c) Find the acceleration vector as a function of time.
   (d) Which way is "up?"
   (e) Find the equation of the plane containing the trajectory.
   (f) If $b = 0$, show that the trajectory is a parabola in the xz-plane.
   (g) Describe the trajectory in the general case.

7. Cross Products and Torque for Cyclists

The effect of a bicyclist's foot on the pedal is measured by the *torque* produced at the center of the gear which is attached to the pedal crank. The physical notion of torque is so closely tied to the mathematical notion of cross product that Feynman introduces them both in the same chapter of his famous lectures on physics (Lecture 20, Volume I in the reference below). In this worksheet, we will use the cross product to analyze pedaling techniques, and we will see how this physical example helps us to understand the mathematics of the cross product.

Three vectors are relevant to our problem:

- **F**: the force applied by the rider's foot to the pedal
- **r**: the "radius" vector from the gear's center G to the point P on the pedal where the force is applied
- **a**: a unit vector along the rotation axis of the gear.

The torque at G produced by the force $\mathbf{F}$ applied at P is the cross product

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}. \tag{4}$$

Since the gear can revolve only around its fixed axis, the effective part of the torque is its component in the direction of **a**, i.e. the scalar product

$$\tau_{\text{eff}} = \boldsymbol{\tau} \cdot \mathbf{a}. \tag{5}$$

Questions

1. Is Equation (4) a mathematical fact, a physical fact, or a definition?

2. Find the cross product of the vectors $(3,5,6)$ and $(2,1,0)$. Relate the result to the area of a geometric figure.

3. How can you use the cross product to determine when three given vectors lie in a plane?

4. Given vectors $\mathbf{A}$ and $\mathbf{B}$, find an identity which relates the four quantities: $||\mathbf{A}||$, $||\mathbf{B}||$, $||\mathbf{A} \times \mathbf{B}||$, and $|\mathbf{A} \cdot \mathbf{B}|$. [**Hint**: Use the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.]

Problems

1. In which direction(s) must the force vector $\mathbf{F}$ point so that the torque vector $\boldsymbol{\tau}$ points in the direction of the axis $\mathbf{a}$?

2. If the magnitude of $\mathbf{F}$ is fixed (by the strength of the cyclist), in which direction should $\mathbf{F}$ point so that the effective torque τ_{eff} is maximized? (Use Equations (4) and (5).)

3. Use the vector identity
$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A}$$
to rewrite the effective torque as the dot product of $\mathbf{F}$ with a certain vector. Show that this leads more quickly to the result of Problem 2.

4. Express τ_{eff} as the volume of a certain parallelepiped. How does this help use choose the optimal direction for $\mathbf{F}$? (This may help more for some students than others!)

5. The vector $\mathbf{r}$ is not quite perpendicular to $\mathbf{a}$ Show that you can replace $\mathbf{r}$ by a vector which *is* perpendicular to $\mathbf{a}$ in such a way that the result of the computation of τ_{eff} is unchanged.

6. Explain the function of bicycle clips and the technique known as "ankling". (Guess what this is if you don't already know.)

Additional problems

In these problems, we will compare the work done by a cyclist using different pedaling techniques. If $\tau_{\text{eff}}(\theta)$ is the effective torque when the pedal crank (the piece of metal connecting the pedal with the gear axis) makes an angle of θ with the vertical, then the work done as the pedal makes half a revolution is the integral

$$\int_0^\pi \tau_{\text{eff}}(\theta)\, d\theta.$$

1. Find $\tau_{\text{eff}}(\theta)$ if the force is always applied downward, with a magnitude F, and then compute the total work done in a half revolution of the pedal.

2. Suppose now that the magnitude of the force is still F, but that it is now applied in the most efficient direction for each value of θ. As in Additional Problem 1, compute $\tau_{\text{eff}}(\theta)$ and the total work done in a half revolution of the pedal.

3. How much is gained by pushing on the pedal in the most effective direction? Can you actually do this on a bicycle?

4. When is it reasonable to compute the work only for a half- revolution of the pedal?

5. Make some other assumptions about the direction of the applied force and compute the work done.

Reference

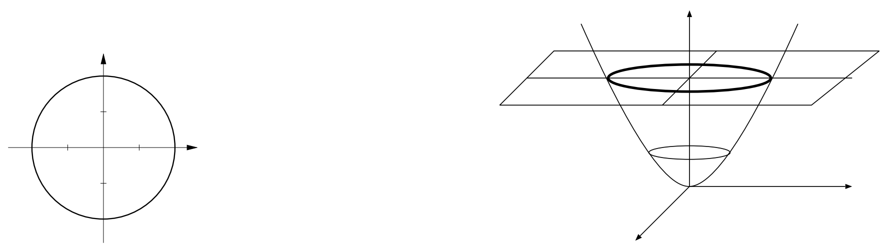
Feynman, Richard P., Leighton, Robert B., and Sands, Matthew. *The Feynman Lectures on Physics*. Addison-Wesley. 1964.

8. Functions of Several Variables

One way to understand the behavior of functions is visually. Here we investigate pictorial ways to represent functions of the form $f\colon \mathbf{R}^n \to \mathbf{R}^m$. A familiar way to represent a function is by its graph. Formally, the *graph of* $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is the set of points $\{(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbf{R}^{n+m} \mid (y_1, \ldots, y_m) = f(x_1, \ldots, x_n)\}$. For most of us, the graph is impossible to visualize if $n + m > 3$.

For functions of the form $f\colon \mathbf{R}^n \to \mathbf{R}$, we have another visualization tool at our disposal. A *level set* of f is the set $\{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid f(x_1, \ldots, x_n) = C\}$ for some constant C. For example, if $f(x, y) = x^2 + y^2$, then the level set $f(x, y) = 4$ is a circle of radius 2 as seen in Figure 5(a). Notice that a level set can be more complicated than a graph. In particular, it does not have to satisfy the vertical line test. A level set of f is actually the "projection" of a certain cross-section of the graph of f to the xy-plane. We see an example of this in Figure 5(b).

The graph of a function can be obtained as the level set of another function by the following trick. The graph of $f(x, y)$ is the level set $g(x, y, z) = 0$ where $g(x, y, z) = f(x, y) - z$. To see that this works, simply note that $g(x, y, z) = 0$ implies that $f(x, y) - z = 0$, so $z = f(x, y)$.



(a) The level set $f(x, y) = 4$ when $f(x, y) = x^2 + y^2$

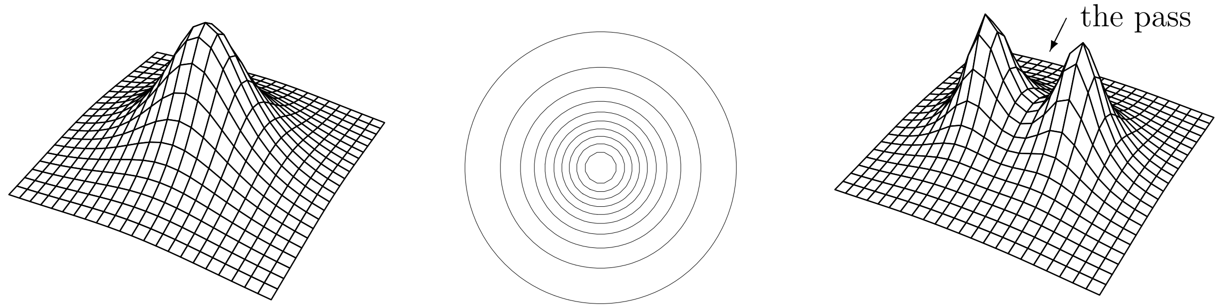
(b) The graph of $f(x, y) = x^2 + y^2$ and the level set $x^2 + y^2 = 4$

Figure 5:

Questions

1. Given functions $f(x, y)$ and $g(x, y)$ and two different numbers c and d, interpret each of the following phrases.
   (a) the intersection of the level sets $f(x, y) = c$ and $g(x, y) = d$.
   (b) the intersection of the level sets $f(x, y) = c$ and $f(x, y) = d$.
   (c) the intersection of graph $z = f(x, y)$ and the graph $z = g(x, y)$.

2. Level sets are used in topographical maps to indicate the contour of the land. For instance, a single mountain is seen topographically in Figure 6(b). Sketch a topographical map of the "mountain range" shown in Figure 6(c). Be sure to draw a level curve that includes the indicated "pass."



(a) A three-dimensional view of a "mountain." (b) A topographical map of the same "mountain." (c) A "mountain range"

Figure 6:

3. Given a point on a level set, is there a unique tangent line at that point?

Problems

1. Sketch the level sets for levels 1, 2, and 3 for each of the following functions.
   (a) $f(x, y, z) = x + y + z$
   (b) $f(x, y, z) = x^2 + y^2/4 + z^2/9$
   (c) $f(x, y, z) = 1/(x^2 + y^2 + z^2)$
   (d) $f(x, y) = xy$
   (e) $f(x, y) = x^2 - y^2$

Additional Problem

1. In Question 2 you drew a topographical map for a mountain range consisting of two mountains. Draw such a map for the "Three Sisters," whose peaks are arranged in a triangle.

9. Limits and Continuity

The subject of limits and continuity is very much the same on $\mathbf{R}^n$ as it is on $\mathbf{R}$. Continuity may be best explained by what it is not. A discontinuous function is one which rips, tears, or punctures the domain when it maps it into the range. In Figure 7(a) we see an example of a discontinuous function,

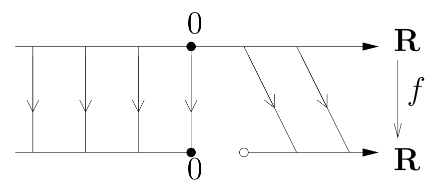
$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x+1 & \text{if } x > 0, \end{cases}$$

which rips the real line at the origin. The graphs of discontinuous functions are characterized by sudden jumps, holes, or shooting off to infinity (vertical asymptotes). The graph of f has a jump discontinuity at 0 as may be seen in Figure 7(b).

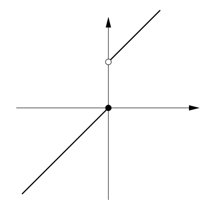
By contrast, continuous functions take points which are near each other to points which are near each other, producing no rips or tears. An example is the function $g(x) = x+1$, seen in Figure 7(c). The graphs of continuous functions are unbroken and without gaps as in Figure 7(d). The main challenge of continuity is to say precisely what we mean by "unbroken" and "without gaps," to formalize the notions of "ripping," "tearing," and "puncturing." This took the best mathematicians many years to get right, so don't despair if it takes you a few tries. Here is the definition of continuity:

> f is *continuous at* $\mathbf{x}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that
>
> $$|\mathbf{y} - \mathbf{x}| < \delta \Longrightarrow |f(\mathbf{y}) - f(\mathbf{x})| < \varepsilon.$$

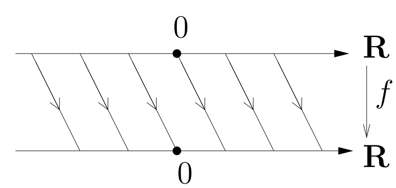
One way to read "$|\mathbf{y} - \mathbf{x}| < \delta$" is "the distance from $\mathbf{y}$ to $\mathbf{x}$ is less than δ." Notice that the definition implicitly requires f($\mathbf{x}$) to be defined; if it isn't then f isn't continuous at $\mathbf{x}$. Your job is to figure out how this definition relates to the intuitive notions discussed above.



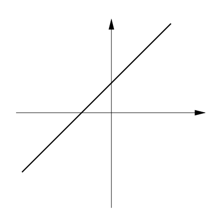
(a) A function which is discontinuous at 0



(b) A graph with a jump discontinuity at 0



(c) A continuous function



(d) A graph of a continuous function

Figure 7:

Roughly speaking, $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{c}$ means that as $\mathbf{x}$ gets close to $\mathbf{a}$, $f(\mathbf{x})$ gets close to $\mathbf{c}$. The technical definition is:

> $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{c}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that
>
> $$0 < |\mathbf{x} - \mathbf{a}| < \delta \Longrightarrow |f(\mathbf{x}) - \mathbf{c}| < \varepsilon.$$

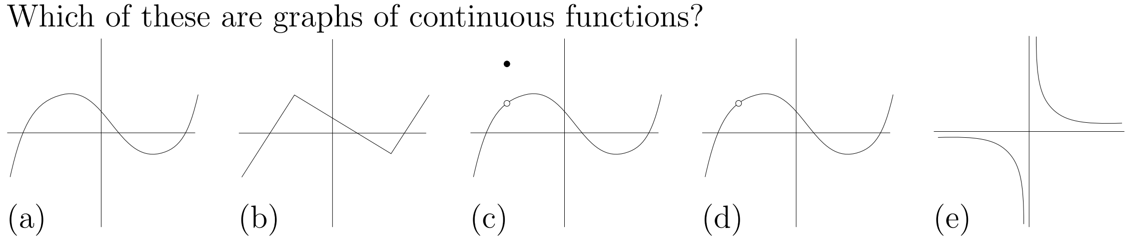
Notice that in the definition of limit, $\mathbf{x} \neq \mathbf{a}$, so that $f(\mathbf{a})$ need not exist for the limit to exist. The definitions of continuity and limit look quite similar, and in fact the definition of continuous function can be equivalently restated in terms of limits as:

$$f \text{ is } \textit{continuous at } \mathbf{a} \text{ if } \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

It is here where we see the main difference when we move from $\mathbf{R}$ to $\mathbf{R}^n$. Namely, in $\mathbf{R}$ it is sufficient to check that both the left-hand and right-hand limits of f exist and are equal. But in $\mathbf{R}^2$ there are an infinite number of paths on which to approach a. For f to be continuous, we must get the same result for $\lim_{x\to a} f(x)$ as we slide x in towards a along any of these paths.

Questions

1. Which of these are graphs of continuous functions?



2. If true, justify. If false, give a counter example.

   (a) The sum of continuous functions is continuous.

   (b) The product of continuous functions is continuous.

   (c) The quotient of continuous functions is continuous.

   (d) The composition of continuous functions is continuous.

Problems

1. Let $f(x, y, z) = 5$. Prove that f is continuous using the ε-δ definition of continuity. This style of proof shows that all constant functions are continuous.

2. Determine whether each of the following limits exist, and if so, find the limit.

$$\text{(a) } \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2}{x^2+y^2}\right) \qquad \text{(b) } \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2}{x^2+y^2}\right) \qquad \text{(c) } \lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}$$

3. Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is continuous and let $g(\mathbf{x}) = -f(\mathbf{x})$. Show that g is continuous. (**Hint:** Question 2 and Problem 1.)

Additional Problems

1. Let $f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ where $f_1, \ldots, f_m$ are real-valued functions. Show that f is continuous if and only if $f_1, \ldots, f_m$ are.

2. Prove the equivalence of the ε-δ and limit based definitions of continuity.

10. Partial Derivatives

Partial derivatives are a multivariable analogue to the single-variable derivative. Suppose that we are given a function $f(x_1, \ldots, x_n)$ and we wish to know the effect of the value of f when we vary just one of the input variables. The rate of change of f as x_i changes is the *partial derivative of f with respect to x_i*:

$$\frac{\partial f}{\partial x_i}(c_1, \ldots, c_n) = \lim_{h \to 0} \frac{f(c_1, \ldots, c_{i-1}, c_i + h, c_{i+1}, \ldots, c_n) - f(c_1, \ldots, c_n)}{h}$$

Other notations for $\frac{\partial f}{\partial x_i}$ are $D_i f$ and f_{x_i}.

Geometrically, $\frac{\partial f}{\partial x_i}$ is the slope of the tangent vector the the graph of f in the plane illustrated in Figure 8.

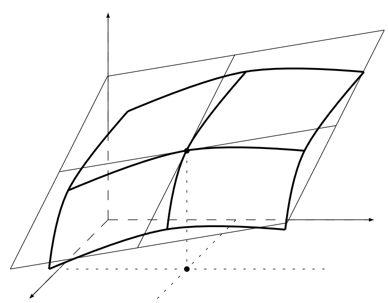


Figure 8: Lines of slope $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

As a practical matter, calculating partial derivatives for a given function is easy: differentiate with respect to one variable as if all the other variables were constants. For example, if $f(x,y) = x^2y + y^3$ then $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2 + 3y^2$. Notice that the partial derivatives are also functions of x and y, so we are free to take partial derivatives of these new functions:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2y \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6y.$$

These higher-order partial derivatives are sometimes written as $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y \partial x}$, or as f_{xx} and f_{xy}. Notice that in our example $f_{xy} = f_{yx}$, i.e. the order in which we take the derivatives doesn't matter. This is no coincidence since Clairaut's Theorem states that if f_{xy} and f_{yx} are continuous then they are equal. This generalizes to higher-order derivatives.

Questions

1. Let $f(x,y) = 5 - x^2 - y^2$.

(a) Find $f_x(1,2)$ and $f_y(1,2)$.

(b) Sketch these two slopes on the graph of $f(x,y)$.

(c) Sketch the level curve through the point $(1,2)$. How might you predict the relative steepness of the two slopes from the direction of the level curve through $(1,2)$?

2. Let $f(x,y) = \begin{cases} 1 & \text{if } xy = 0 \\ 0 & \text{if } xy \neq 0. \end{cases}$

(a) For which points (x,y) does $f_x(x,y)$ exist?

(b) For which points (x,y) does $f_y(x,y)$ exist?

(c) At which points is f continuous?

3. The volume of a rectangular box of width x, height y and depth z is given by $V = xyz$.

(a) Calculate $\dfrac{\partial V}{\partial x}$, $\dfrac{\partial V}{\partial y}$, and $\dfrac{\partial V}{\partial z}$.

(b) Note that each partial derivative gives the area of a side of the box. Draw a picture illustrating why the rate of change of volume with respect to x is equal to the area of one side of the box.

Problems

1. Let $f(x,y) = \sqrt{x^2 + y^2}$.

(a) Graph $z = f(x,y)$.

(b) Based on your graph, decide if $f(x,y)$ has continuous partial derivatives everywhere.

(c) Verify (b).

2. Let $f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Compute $\dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$ for $(x,y) \neq (0,0)$.

(b) Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.
(**Note:** You will need to use t definition of a partial derivative as a limit.)

(c) Does f have continuous partial derivatives?

11. Tangent Planes and Differentials

In this section we discuss tangent planes to graphs and the related algebraic objects called differentials. Let $f(x, y)$ be a function with partial derivatives that we can calculate and suppose that we wish to understand how f varies as we perturb (x, y) about a point (x_0, y_0). We have already seen that $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ measure the rates of change of f with respect to x and y, so it is plausible to approximate the change in f as we move from (x_0, y_0) to (x, y) by

$$\Delta f \approx \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y. \tag{6}$$

The right hand side of Equation 6 is called the *best linear approximation to* Δf We encode the best linear approximation algebraically by the *differential* of f,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

To understand the best linear approximation geometrically we first rewrite it as

$$f(x, y) \approx \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + f(x_0, y_0).$$

We can then graph both the surface $z = f(x, y)$ and its approximation, $z = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) + f(x_0, y_0)$. The second graph is the *tangent plane to the surface* $z = f(x, y)$ *at the point* (x_0, y_0). In this context we can see that Δf measures the change in the height of the surface as we move from (x_0, y_0) to (x, y), while df measures the change in the height of the tangent plane.

Questions

1. Recall that in single-variable calculus if x_0 is a local extremum (maximum or minimum) of $f(x)$ then the tangent line to the graph of f at x_0 is horizontal, so $\frac{df}{dx}(x_0) = 0$.
   (a) Can you find a similar relation between the local minimum or maximum of a function of two variables and the tangent plane to its graph?
   (b) Use this relation to find the local extrema of the function $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.
   (c) Verify that your answer in (b) is correct by completing the squares in $f(x, y)$ and identifying the quadric $f(x, y)$.

Problems

1. Find the point in $\mathbf{R}^3$ where the z-axis intersects the plane that is tangent to the graph of $z = e^{x-y}$ at $(1, 1, 1)$.

2. Biggles the Surveyor is determining the distance between two points along a coast line as in Figure 9. Biggles finds that the line segment connecting the two points forms the

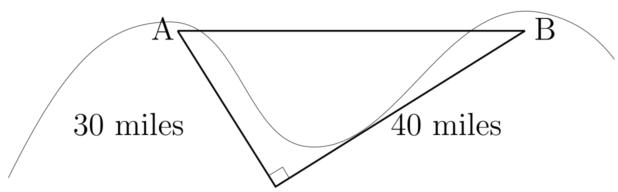


Figure 9: A coast line

hypotenuse of a right triangle whose legs are of lengths 30 and 40 miles, plus or minus 0.1 miles.

(a) What is the distance between the two points on the coast?

(b) Use differentials to estimate the error in the distance computed in part (a).

(c) Compute the maximum possible error in the distance computed in part (a) by substituting extreme values for the base and altitude of the triangle.

(d) In general, can you find the maximum possible error in the output of a function by plugging in extreme values for the variables?

12. The Chain Rule

The chain rule is one of the most important practical tools of calculus since it expresses the derivative of a complicated function in terms of the derivatives of simpler constituent functions. Recall from single-variable calculus that

$$\frac{d}{dt}f(x(t)) = \frac{df}{dx}(x(t)) \cdot \frac{dx}{dt}(t)$$

which can be interpreted as

> (The change in f due to a change in t) = (the change in f due to the change in x) $\cdot$ (the change in x due to the change in t).

Suppose that we are instead given $g(x, y)$, a differentiable function of two variables, and that each variable is a function of time, and we wish to compute $\frac{dg}{dt}$. For example, let $g(x, y) = \sin(x + y)$, $x(t) = t$, and $y(t) = 2t$. One way to proceed is to substitute t and $2t$ for x and y and then use the single-variable chain rule: $g(t) = \sin(3t)$ so $\frac{dg}{dt} = 3\cos(3t)$. Another is to first use the multivariable chain rule and then make the substitutions: $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = \cos(x + y)(1) + \cos(x + y)(2) = 3\cos(x + y) = 3\cos(3t)$. We can interpret the chain rule as

> (The change in g due to the change in t) = (the change in g due to the change in x) $\cdot$ (the change in x due to the change in t) + (the change in g due to the change in y) $\cdot$ (the change in y due to the change in t).

The relations of this awkward phrase can be seen more clearly in the *dependency diagram* in Figure 10(a). Each summand of the chain rule is obtained by following a path from the top of the diagram to the bottom.

In general if $h(x_1, \ldots, x_n)$ is a differentiable function and each x_i itself a function of n variables $x_i = x_i(t_1, \ldots, t_n)$ then

$$\frac{\partial h}{\partial t_j} = \frac{\partial h}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial h}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial h}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

which we is summarized by the dependency diagram in Figure 10(b).

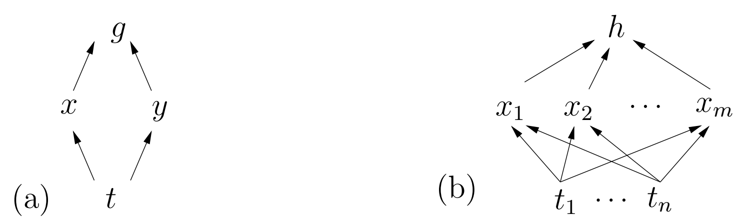


Figure 10: Dependency diagrams

Questions

1. What information does the dependency diagram in Figure 10(a) encapsulate for $\frac{dg}{dt}$?

2. Suppose that $z = f(x, y)$, $x = g(u, v)$, $y = h(u, v)$.

   (a) Give expressions for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

   (b) Suppose as well that $u = \phi(t)$ and $v = \psi(t)$. Give expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

   (c) Now give an expression for $\frac{dz}{dt}$. Draw the appropriate dependency diagram.

Problems

1. Suppose that the temperature in a lake is given by $T(x, y) = x^2e^y - xy^3$ and that you are swimming in a circle at constant speed: $x(t) = \cos t$ and $y(t) = \sin t$. Find $\frac{dT}{dt}$ in two ways:

   (a) By the chain rule

   (b) By finding T explicitly as a function of t and differentiating.

2. Suppose that a particle in $\mathbf{R}^3$ stays on the surface $xy + yz + xz = 0$. If the particle is at the point $(2, -1, 2)$ and starts moving along the surface parallel to the xz-plane, how fast is z changing with respect to x?

13. Directional Derivatives and the Gradient Vector

Given a function $f(x_1, \ldots, x_m)$, the directional derivative is the rate of increase of the function as the point in the domain moves in some direction. For example, if that direction were along the y-axis (with y increasing), then the directional derivative would be the partial derivative of f with respect to y. To find a directional derivative, we must specify a starting point in the domain and a direction as a vector from this point.

If $\mathbf{x} \in \mathbf{R}^m$ is the point and $\mathbf{u} \in \mathbf{R}^m$ is the direction, the definition of the directional derivative is

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{|t\mathbf{u}|}.$$

The gradient is an m-dimensional vector ∇f that takes different values at different points of the domain of f according to the equation

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_m} \right\rangle.$$

The gradient has two fundamental geometric properties:

1. At every point of the domain of f, it points in the direction of maximal increase of f.
2. At every point of the domain of f, it is perpendicular to the level surface of f passing through that point.

Given any surface in $\mathbf{R}^3$, in order to find the tangent plane at a point, simply write the surface as a level surface of a function $f(x, y, z)$; then ∇f at the point is perpendicular to the surface, and hence is the normal vector to the tangent plane. Then the point and the normal vector determine the tangent plane. For example, if the surface is the graph of a function $z = g(x, y)$, then it is the level surface $f(x, y, z) = z - g(x, y) = 0$.

The gradient is used to compute directional derivatives via the following formula

$$D_{\mathbf{u}} f(\mathbf{x}) = \nabla f \cdot \frac{\mathbf{u}}{|\mathbf{u}|}.$$

Questions

1. True or False? The gradient of the function $f(x, y) = x^2 + y^2$ at the point $(1, 1)$ is $2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
2. Suppose that for $f(x, y)$ the level curve through a certain point forms an "$\times$". How can ∇f be perpendicular to all of the level curve segments at this point?
3. A dye is used to detect the presence of a certain chemical — higher concentrations of the chemical make the dye grow darker. For each slide treated with the dye seen in Figure 11 below:

(a) Draw in curves of constant concentration.

(b) Draw in gradient vectors pointing in the direction of increasing concentration.

(c) Indicate points where the gradient is zero.

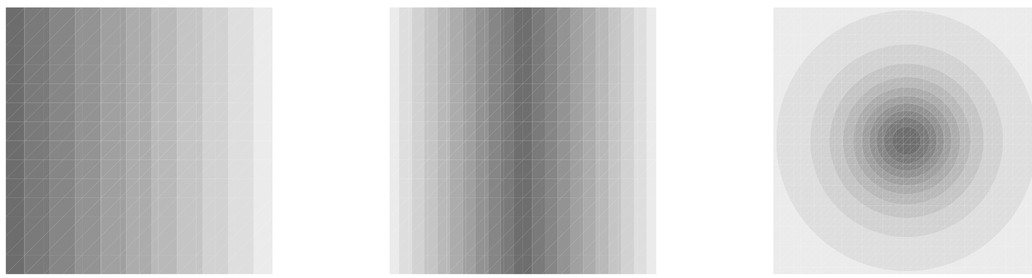


Figure 11: Slides treated with a special dye.

Problems

1. Let $f(x, y) = x^2 + y^2 - xy$.

   (a) Compute $\nabla f(1, 3)$.

   (b) Compute the directional derivative of f at $(1, 3)$ in the direction of $\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$.

2. The sphere $x^2 + y^2 + z^2 = 6$ and the ellipsoid $x^2 + 3y^2 + 2z^2 = 9$ intersect at the point $(2, 1, 1)$. Find the angle between their tangent planes at this point.

3. Suppose the function $f \colon \mathbf{R}^m \to \mathbf{R}$ is *even*, i.e. $f(\mathbf{x}) = f(-\mathbf{x})$ for every $\mathbf{x} \in \mathbf{R}^m$, and suppose f is differentiable. Find ∇f at the origin.

14. Maximum and Minimum Values

Recall from the preceding section that the gradient of f at a point of the domain points in the direction of maximal increase of f. If f has a local maximum at some point, and the gradient exists at that point, then the gradient must equal $\mathbf{0}$ there—there cannot be any direction of positive maximal increase from this point, since it is a local maximum. An analogous argument can be made for a local minimum.

In single-variable calculus, the candidates for maxima and minima were points at which the derivative equaled zero or did not exist or at the endpoints of an interval. In multivariable calculus, the candidates for maxima and minima are points at which the gradient equals the zero vector or does not exist. This is a sensible generalization since the gradient of a single-variable function is just the derivative. We will deal later with the multivariable equivalent of endpoints.

We classify critical points of a function of two variables, f, as local maxima, local minima, or saddle points with the following second derivative test:

Suppose that (x_0, y_0) is a critical point, meaning $\nabla f(x_0, y_0) = \mathbf{0}$.

1. If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{yx}(x_0, y_0))^2 = 0$ then the second derivative test gives no information.
2. If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{yx}(x_0, y_0))^2 < 0$ then (x_0, y_0) is a saddle point.
3. If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{yx}(x_0, y_0))^2 > 0$ then:
   (a) If $f_{xx}(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum.
   (b) If $f_{xx}(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum.

Questions

1. Suppose that $f(x, y)$ has a critical point at $(0, 0)$ and the slices of the graph of f lying in the planes $x = 0$ and $y = 0$ are both concave upward. Does it follow that f has a local minimum at the origin?

Problems

1. Find and classify the critical points of the following functions:

   (a) $f(x, y) = x^5 + y^4 - 5x - 32y + 81$

   (b) $f(x, y) = x^3 + y^3 + 3xy - 27$

   (c) $f(x, y) = x^2y + 3xy - 3x^2 - 4x + 2y + 1$

2. In (a) and (b), work with your group to draw the graph of a continuous function in two variables which has the desired counter-intuitive properties.

   (a) Draw the graph of a function with two critical points, both of which are maxima.

   (b) Draw the graph of a function with exactly one critical point which is a local maximum but not a global maximum.

Additional Problems

1. (a) Use the second derivative test to show that a box of fixed surface area has maximum volume when it is a cube.

   (b) Similarly, use the second derivative test to show that a box of fixed volume has minimum surface area when it is a cube.

   (c) Does the answer to (a) imply the answer to (b)? How about *vice-versa*?

15. Lagrange Multipliers

Whereas the last section dealt with finding *unconstrained* extrema, this section deals with finding *constrained* extrema. The only difference is the domain of the function: constrained extrema are local maxima or minima of a function whose domain is not all of $\mathbf{R}^m$, but some lower-dimensional object in $\mathbf{R}^m$, such as a spherical surface in $\mathbf{R}^3$, or a parabolic curve in $\mathbf{R}^2$.

Suppose that we want to find the extrema of a function $f(x, y, z)$ whose domain is a two-dimensional surface S. If S can be written as a level surface $g(x, y, z) = c$ for a constant c, then to find possible extrema, we solve the equations $\nabla f = \lambda \nabla g$, $g(x, y, z) = c$, $\lambda \neq 0$. This is really four equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}, \quad \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z}, \quad \text{and} \quad g(x, y, z) = c$$

in four unknowns: x, y, z, and λ where $\lambda \neq 0$. The value we get for λ may have no physical significance (although it often does), but when we get a point (x_0, y_0, z_0) that (along with some λ_0) simultaneously satisfies all four equations, this point is one of our candidates for local maximum or local minimum.

Questions

1. Let $f(x, y) = x$ and $g(x, y) = x^2 + y^2$.

   (a) Compute ∇f and ∇g.

   (b) Draw the unit circle $g(x, y) = 1$ and plot the vectors ∇f and ∇g on the unit circle at the points with polar coordinates $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$.

   (c) Suppose that an ant is constrained to crawl on the unit circle and that it wants to crawl in the direction that will make f increase. At each point in your diagram, indicated whether the ant crawls clockwise or counterclockwise.

   (d) Which points on the unit circle maximize f? Which points minimize it? Is $\nabla f = \lambda \nabla g$ at these points?

2. How can we find the local extrema of a function on the interior or perimeter of a triangle in the plane?

3. How would we solve the Lagrange multiplier equations if $f(x, y, z) = g(x, y, z)$?

Problems

1. Find the point (x, y) with the largest y value lying on the curve in $\mathbf{R}^2$ whose equation is $y^2 = x - 2x^2 y$.

2. What is the volume of the largest box with edges parallel to the coordinate axes that fits inside the ellipsoid
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1?$$

3. Show that a triangle with fixed perimeter has maximal area if it is equilateral. (**Hint:** According to Heron's formula, the area of a triangle with side lengths x, y, and z is $A=\sqrt{s(s-x)(s-y)(s-z)}$, where $s=(x+y+z)/2$.)

16. Double Integrals over Rectangles

As an approach to understanding double integrals, it is useful to consider the following problem. Let $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ be a rectangular metal plate and let $f(x, y)$ give the density of the metal as a function of position. What is the mass of the plate? To get an approximate answer, we divide the plate into many small rectangles as in Figure 12. If (x_i^*, y_j^*) is an arbitrary point in subrectangle R_{ij} and $\Delta A_{ij} = \Delta x_i \Delta y_j$ is the area of subrectangle R_{ij} then

$$\text{mass of } R_{ij} \approx \big(\text{density at } (x_i^*, y_j^*)\big)\,(\text{area of } R_{ij}) = f(x_i^*, y_j^*)\,\Delta A_{ij}.$$

Summing up the approximate masses of all the subrectangles, we find that the mass of R is roughly $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A_{ij}$. Presumably we get a more accurate estimate if we use a finer partition of R. This motivates the following definition

The *double integral of f over the rectangle R* is

$$\iint\limits_R f\, dA = \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A_{ij}$$

where $\|P\|$ is the length of the longest diagonal of all the subrectangles of R in a given partition.

R_{11}	R_{21}	$\cdots$	R_{m1}
R_{12}	R_{22}	$\cdots$	R_{m2}
$\vdots$	$\vdots$		$\vdots$
R_{1n}	R_{2n}	$\cdots$	R_{mn}

Figure 12:

This definition is good for proving basic properties of integration, such as $\iint_R (f+g)\, dA = \iint_R f\, dA + \iint_R g\, dA$, but it is unworkable for most practical purposes. Our practical tool is Fubini's theorem which says that a double integral may be evaluated by doing two single-variable integrals

If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

This is frequently abbreviated to $\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dx\, dy$ and the method is called *iterated integration*. Intuitively, Fubini's theorem is saying that to find the mass of a metal plate we should divide it into vertical (or horizontal) strips, use integration to find the mass of a typical strip, and then integrate again to sum the masses of all of the strips.

Questions

1. Suppose that $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ is a rectangle with area P and that $f(x, y) \geq Q$ where Q is a constant.

   (a) Use the definition of double integration to show that $\iint_R f(x, y)\,dA \geq PQ$.

   (b) State the analogous result if $f(x, y) \leq Q$.

2. Let $f(x, y) = \cos(x^2 + y^2)$ and
   let R be the rectangle $\{(x, y) \mid -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}$.

   (a) Prove that $-\pi^2 \leq \iint_R f(x, y)\,dA \leq \pi^2$.

   (b) Write down any integral estimate of $\iint_R f(x, y)\,dA$ with four or more subrectangles. Specify what subrectangles you are using and which (x_i^*, y_j^*) you are choosing.

Problems

1. Let $R = \{(x, y) \mid 1 \leq x \leq 4, -2 \leq y \leq 3\}$. Evaluate $\iint_R (x^2 - 2xy^2 + y^3)\,dA$.

2. Let $R = \{(x, y) \mid 1 \leq x \leq 3, 1 \leq y \leq 2\}$. Evaluate $\iint_R \frac{x}{y}\,dA$.

3. Evaluate $\displaystyle\int_{-1}^{1} \left(\int_0^2 \sin y e^{\cos x}\,dx \right) dy$
   (**Hint:** Separate this into the product of two integrals.)

Additional Problem

1. (a) Graph the function $z = \cos\sqrt{1 - y^2}$ over the rectangle $[-\pi/2, \pi/2] \times [-1, 1]$.
   (**Hint:** Imagine a bear in a large sleeping bag.)

   (b) What is the volume of the region trapped between the xy-plane and this graph?

17. Double Integrals over General Regions

In this section we extend the domain of integration of a double integral from rectangles to more general regions of the plane. Suppose D is a metal plate whose region is pictured in Figure 13(a) and $f(x, y)$ is a density function on D. What is the mass of D? Informally, we proceed as in the previous section: partition D into subrectangles, evaluate the density function inside each subrectangle, multiply by the area of each subrectangle to obtain an approximation to its mass, and sum over all subrectangles to approximate the mass of D. Passing to the limit as the partition gets finer gives the mass of D.

(a) (b)

Figure 13: A curvy metal plate

The problem is that the curved edges of D make it impossible to divide precisely into rectangles. To deal with this, we choose a rectangle R enclosing D, as in Figure 13(b), and define a function $g(x, y)$ extending $f(x, y)$ by

$$g(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D. \end{cases}$$

We define the *integral of* $f(x, y)$ *over* D to be $\iint\limits_D f(x,y)\,dA \overset{def}{=} \iint\limits_R g(x,y)\,dA$. Since R is a rectangle, we can apply the definitions and theorems of the previous section to compute $\iint\limits_R g(x,y)\,dA$.

It may seem that the scheme outlined above only complicates things, but by following it carefully we can actually compute some new integrals. For example, let D be the triangle bounded by the x-axis, the y-axis, and the line $y = x$. Let $f(x, y) = xy$ and suppose we

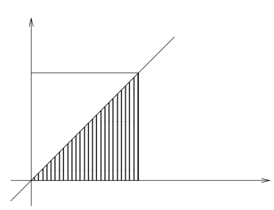


Figure 14: A triangle enclosed by a unit square

want to compute $\iint\limits_D f(x,y)\,dA$. Let R be the unit square enclosing D as in Figure 14 and let $g(x,y)$ extend $f(x,y)$ by

$$g(x,y) = \begin{cases} xy & \text{if } y \le x \\ 0 & \text{if } y > x. \end{cases}$$

Then

$$\begin{aligned}
\iint\limits_D f(x,y)\,dA &= \iint\limits_R g(x,y)\,dA && \text{by definition} \\
&= \int_0^1 \left(\int_0^1 g(x,y)dy\right)\,dx && \text{by Fubini} \\
&= \int_0^1 \left(\int_0^x g(x,y)\,dy + \int_{y=x}^1 g(x,y)\,dy\right)\,dx && \text{by single variable calculus} \\
&= \int_0^1 \left(\int_0^x xy\,dy + \int_x^1 0\,dy\right)\,dx && \text{by definition of } g(x,y) \\
&= \int_0^1 \tfrac{1}{2}x^3\,dx && \text{by single variable calculus} \\
&= \tfrac{1}{8}
\end{aligned}$$

Typically we abbreviate this calculation by never mentioning R or g:

$$\iint\limits_D f(x,y)\,dA = \int_0^1 \int_0^x xy\,dy\,dx = \int_0^1 x\left[\frac{1}{2}y^2\right]_0^x\,dx = \int_0^1 \frac{1}{2}x^3\,dx = \frac{1}{8}.$$

Questions

1. Let D be a region in the plane. What geometric quantity is computed by $\iint\limits_D dA$?

2. How would you compute the area of D if it were drawn on graph paper? Why does a finer grid give a better approximation to the area? (**Hint:** Discuss the boxes which touch the edge of D.)

Problems

1. Let D be the region of $\mathbf{R}^2$ bounded by the lines $y = 1$, $y = 2$, $x = 0$ and $x = y$. Compute $\iint\limits_D x^2y^2\,dA$.

2. Let E be the region of $\mathbf{R}^3$ bounded by the plane $z = 0$, the cylinder $y = x^2$, the cylinder $x = y^2$, and the graph of $z = xy$. Find the volume of E.

3. Rewrite the following double integrals with the order of integration reversed.

   (a) $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$

   (b) $\int_{1/3}^{2/3} \int_{y^2}^{\sqrt{y}} f(x,y)\,dx\,dy$

Additional Problems

1. The graph $y = f(x)$, $a \le x \le b$ is rotated about the x-axis (assume $f(x) \ge 0$).

   (a) Draw the figure of revolution in xyz-space for $f(x) = 2 + \cos x$, $\frac{-3\pi}{2} \le x \le \frac{3\pi}{2}$.

   (b) For general $f(x)$, describe the region of the xy plane enclosed by the figure.

   (c) Derive the formula for the volume of the figure:

$$V = 2\int_a^b \int_{-f(x)}^{f(x)} \sqrt{[f(x)]^2 - y^2}\, dy\, dx.$$

   (d) Can you reconcile this ugly formula with the fact that we could already do this problem at the end of Math 1A?

18. Double Integrals in Polar Coordinates

Using polar coordinates often simplifies integrals in problems exhibiting circular symmetry. Let us again consider the problem of finding the mass of a metal plate D given a density function $f(r,\theta)$ on it. As before, we approximate the mass of the plate by (1) partitioning it into small pieces, (2) approximating the mass of each piece, and (3) summing the approximate masses. In step (1), Cartesian coordinates lend themselves most readily to partitioning the region into rectangular pieces. Polar coordinates lend themselves to a different partition, illustrated in Figure 15. The edges of the *polar rectangle* in Figure 15 are of lengths Δr

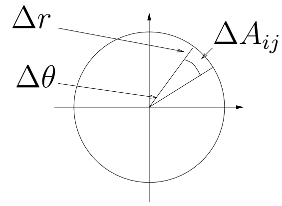


Figure 15: A typical polar area element

and $r\Delta\theta$. For small Δr and $\Delta\theta$ the polar rectangle is nearly a rectangle, so we approximate the area by $\Delta A_{ij} \approx r\,\Delta r\,\Delta\theta$. Now if $(r^*_{ij}, \theta^*_{ij})$ is a point inside the rectangle then the mass of the rectangle is given by $\Delta m_{ij} \approx f(r^*_{ij}, \theta^*_{ij})\Delta A_{ij} = f(r^*_{ij}, \theta^*_{ij}) r \Delta r \Delta\theta$. Summing the mass elements and passing to the limit of arbitrarily fine partitions gives the mass of the plate $\iint_D f(r,\theta)\,r\,dr\,d\theta$.

The formula to change from a Cartesian integral to a polar integral is

$$\iint_D g(x,y)\,dxdy = \iint_R g(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where D and R are corresponding regions in xy- and $r\theta$-space. A good method to find the polar limits of integration is to draw a careful picture of the domain of integration and work from there.

Questions

1. Consider the polar rectangle of Figure 15. Why is the outer edge of length $r\,\Delta\theta$ instead of just $\Delta\theta$? Draw a picture illustrating how increasing r increases the length of that edge.
2. Let $A = \{(x,y) \mid -a \le x \le a, -a \le y \le a\}$ and $B = \{(x,y) \mid x^2 + y^2 \le a^2\}$ where a is a positive constant. Consider the integrals $\iint_A 1\,dA$ and $\iint_B 1\,dA$.
   (a) Find an easy coordinate system to compute each integral and do the computation.
   (b) For each integral, what makes one coordinate system a better choice than the other?

Problems

1. By changing to polar coordinates, compute $\int_0^1 \int_y^{\sqrt{y}} \sqrt{x^2+y^2}\,dx\,dy$.

2. Compute the volume of a right circular cone with base radius a and height h using polar coordinates.

3. (a) Show that $(\int_a^b e^{-x^2}\,dx)^2 = \int_a^b \int_a^b e^{-(x^2+y^2)}\,dx\,dy$.

   (b) Let $D = \{(x,y) \mid -\infty < x < \infty, -\infty < y < \infty\}$ be the entire plane. Compute $\iint\limits_D e^{-(x^2+y^2)}\,dx\,dy$.

   (c) What is $\int_{-\infty}^{\infty} e^{-x^2}\,dx$?

   Note: We could never do this using the elementary techniques of single-variable calculus since e^{-x^2} has no closed form indefinite integral.

19. Applications of Double Integrals

As we have seen, if a function represents density then the integral of the function over some region gives the mass of the region. Integration can compute other interesting physical quantities. For example, if $f(x, y)$ gives the *charge density* on a metal plate R then $\iint_R f\, dA$ computes the net charge on the plate taking into account both positively and negatively charged regions.

Another application is *center of mass*. Newton, after postulating his universal law of gravitation between particles, sought to show that bodies in space whose densities depend only on distance from their centers (such as stars and planets) attract each other as if they were point masses located at their centers of mass. The general formula for the center of mass of an object B in $\mathbf{R}^n$ is as follows. Let

$$\overline{x_i} = \frac{1}{m} \iint \cdots \int_B x_i \rho(x_1, x_2, \ldots, x_n)\, dA,$$

where ρ is density and m is the mass of B. Then $(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n})$ is the center of mass.

Also of physical importance is *moment of inertia*. Just as mass (m) tells us how much energy it takes to move an object at speed v by the formula $E = \frac{1}{2}mv^2$, the moment of inertia (I) tells us how much energy it takes to spin an object around a given axis at angular speed ω by the formula $E = \frac{1}{2}I\omega^2$. The moment of inertia depends on what axis of rotation one is considering — it may be harder to spin a body around one line in space than another (see Figure 16). Let $\xi(x_1, \ldots, x_n)$ be the distance in $\mathbf{R}^n$ from the axis of rotation to the

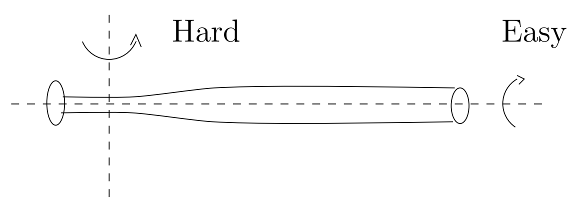


Figure 16: It is harder to swing a bat than to spin it around the long axis

point $(x_1, \ldots, x_n)$ and let $\rho(x_1, \ldots, x_n)$ be the density. Then the moment of inertia of the body B about the axis of rotation is

$$I = \iint \cdots \int_B \xi^2(x_1, \ldots, x_n)\rho(x_1, \ldots, x_n)\, dA$$

Problems

1. Suppose that a unit disk in $\mathbf{R}^2$ has density proportional to the distance from its center. Find a formula for its mass.

2. Suppose that the region if $\mathbf{R}^2$ bounded by the parabolas $y = x^2$ and $x = y^2$ has constant density. Compute its center of mass.

3. Find a formula for the moment of inertia of a unit disk in $\mathbf{R}^2$ about its central point, assuming
   (a) the disk has unit density.
   (b) the disk has density proportional to the distance from its center
4. Suppose that a square of unit density has the same mass as the disk of Problem 3(a) above.
   (a) What are the dimensions of the square?
   (b) Determine the moment of inertia about its center. Is it harder to spin a square or a disk?

Questions

1. Here we derive the formula $E = \frac{1}{2}I\omega^2$ from the formula $E = \frac{1}{2}mv^2$. Assume that a lamina B is spinning around an axis with angular speed ω as in Figure 17.

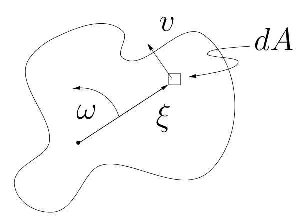


Figure 17: A rotating lamina and a typical piece at distance ξ from the center of rotation

   (a) Consider a small piece of the lamina at distance ξ from the axis of rotation. What is the speed v of this piece as a function of ξ and ω?
   (b) Suppose this mass element has area dA. Write an expression for the mass dm of the piece.
   (c) The kinetic energy of a moving body is given by $E = \frac{1}{2}mv^2$. Combine parts (a) and (b) to find the kinetic energy dE of the mass element in terms of ω, ξ and dA.
   (d) The energy necessary to spin the entire lamina is the sum of the kinetic energies of all the mass elements when spinning; i.e. $E = \iint_B dE$. Use your answer from part (c) to derive the formula $E = \frac{1}{2}I\omega^2$.
2. Which shape lamina of mass m and unit density has the smallest moment of inertia? Why?

20. Surface Area

In order to find the surface area of the graph of a function $f : D \to \mathbf{R}$ where D is some subset of $\mathbf{R}^2$, we "plate" the surface with quadrilaterals as in Figure 18 below. If the total

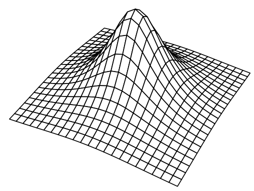


Figure 18: A surface plated with quadrilaterals

area of these plates approaches a limit as the partition norm approaches zero, then we define the surface area to be this limit.

It can be shown that for $f(x, y)$ sufficiently differentiable, the area of the surface $S = \{(x, y, f(x, y)) \in \mathbf{R}^3 \mid (x, y) \in D \subset \mathbf{R}^2\}$ is given by

$$\text{area}(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA. \qquad (7)$$

Questions

1. Let $R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ be a unit square and let P be a parallelogram obtained by projecting R upward onto the plane $z = m_1 x + m_2 y$ as in Figure 19.

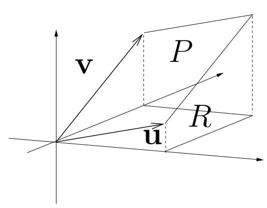


Figure 19: A parallelogram floating above a unit square

   (a) Write $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of m_1 and m_2.
   (b) Write the vectors $\mathbf{u}$ and $\mathbf{v}$ in terms of m_1 and m_2.
   (c) Compute the area of P using $\mathbf{u}$ and $\mathbf{v}$.

2. Let $f(x)$ be a continuously differentiable function, and define $g(x, y) = f(x)$. Let $R = \{(x, y) \in \mathbf{R}^2 \mid a \le x \le b, c \le y \le d\}$, where a, b, c and d are constants. Prove that the surface area of the graph of g over R equals $(d - c)$ times the arc length of the graph of $f(x)$ over $a \le x \le b$. Illustrate.

Problems

1. Let $f(x, y) = \ln \cos x + \frac{2}{3}y^{3/2}$ and let $D = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1$ and $100 - 25x - \sec^2 x \leq y \leq 100 - \sec^2 x$. Find the area of the graph of f over D.

2. Suppose two circular cylinders of equal radii intersect at right angles. Find the total surface area of the solid contained within both cylinders in terms of the cylinders common radii.

21. Triple Integrals in Cartesian, Spherical, and Cylindrical Coordinates

Understanding the integral of $f(x,y,z)$ over a region $E \subset \mathbf{R}^3$ is a simple generalization of earlier concepts: we regard $f(x,y,z)$ as a density function on E and $\iiint_E f\,dV$ as the mass of E. The practical difficulties of finding the limits of integration are greater. As an example, suppose we wish to integrate the function $f(x,y,z)=1$ over the region E bounded by the planes $x=0$, $y=0$, $z=0$, and $x+y+z=10$; i.e. we wish to find the volume of E. We proceed by partitioning E into many parallel lamina and dividing each lamina into many parallel fibers. A typical lamina and a single fiber are seen in Figure 20. The height of

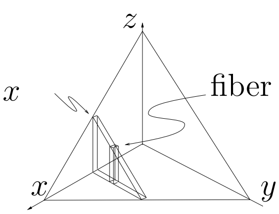


Figure 20: A typical lamina and fiber

the fiber at (x,y) is given by $\int_0^{10-x-y} 1\,dz$ and the area of the lamina at x is obtained by integrating the height of its fibers along its length; i.e. $\int_0^{10-x}\int_0^{10-x-y} 1\,dz\,dy$. Finally, the volume of E is obtained by integrating the area of the lamina at x with respect to x:

$$\text{volume}(E) = \int_0^{10}\int_0^{10-x}\int_0^{10-x-y} 1\,dz\,dy\,dx.$$

Drawing a careful picture of the region of integration, a typical lamina, and a typical fiber makes finding the limits of integration much easier.

Just as some double integrals are easier in polar coordinates, some triple integrals are easier in spherical or cylindrical coordinates. To integrate in these alternate coordinate systems we need to know the formula for a typical volume element. One way to determine this is to draw a good picture. For reference, the formulas for dV as well as the conversion to Cartesian coordinates for each coordinate system appears below.

Spherical

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ dV &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Cylindrical

$$\begin{aligned} x &= r\sin\theta \\ y &= r\cos\theta \\ z &= z \\ dV &= r\,dr\,d\theta\,dz \end{aligned}$$

Problems

1. Set up and evaluate a triple integral giving the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Does your answer give the volume of the unit sphere when $a=b=c=1$?

2. Fill in the appropriate limits of integration for the different integration orders:

$$\int_0^2 \int_0^x \int_{-\sqrt{4-2x}}^{\sqrt{4-2x}} f(x,y,z)\,dy\,dz\,dx = \int_?^? \int_?^? \int_?^? f(x,y,z)\,dz\,dx\,dy = \int_?^? \int_?^? \int_?^? f(x,y,z)\,dx\,dz\,dy$$

3. Compute

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

by changing to spherical coordinates.

4. Suppose the density of a three-dimensional ball is proportional to the distance from the surface. Find a formula for its mass.

5. Suppose a cone of height h and base radius r has density equal to the distance from its base. Find its center of mass.

6. Compute the moment of inertia of:

   (a) a sphere of radius a of unit density about its diameter.

   (b) a cylinder of height h and radius a of unit density about its central axis.

22. Change of Variable in Multiple Integrals

When working integrals, it is wise to choose a coordinate system that fits the problem; e.g. polar coordinates are a good choice for integrating over disks. Once we choose a coordinate system we must figure out the area form for that system. For example, when switching from Cartesian to polar coordinates we must change the form of the area element from $dx\,dy$ to $r\,dr\,d\theta$. To determine that $r\,dr\,d\theta$ is the correct formula we drew a careful picture of a typical area element and noted how the edges of the element varied with r and θ. With more complex coordinate systems, such as spherical coordinates, this procedure becomes more difficult and the diagrams more intricate. Fortunately, there is a method to find the area form by calculation.

Suppose we are integrating a function over a region of n-space with the function and the region of integration given in terms of one set of variables $x_1, \ldots, x_n$, and that we want to change to another set of variables $y_1, \ldots, y_n$. Then

$$\iint \cdots \int_A f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n =$$
$$\iint \cdots \int_B f(x_1(y_1, \ldots, y_n), \ldots, x_n(y_1, \ldots, y_n)) \left| \frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)} \right| dy_1 \ldots dy_n,$$

where A is a region of integration in x_i-space, B is the corresponding region in y_i-space, and $\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}$ is the Jacobian of the transformation. It is important that there be a one-to-one correspondence between the points of A and the points of B under the change of variables, and that the x_i's are differentiable functions of the y_i's.

It should be remarked that we have given no general procedure for finding an appropriate coordinate system to fit a problem. We have only summarized how to find the area form given a coordinate system.

Questions

1. Let $x = 2u$ and $y = 3v$. Let R be a rectangle in the uv-plane with lower left-hand corner (u, v), width Δu and height Δv as in Figure 21.

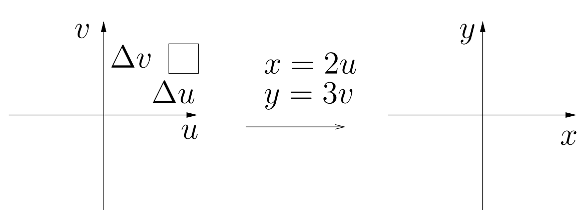


Figure 21: A map from uv-space to xy-space

   (a) Draw the image of R in the xy-plane and compute its area in terms of Δu and Δv.

(b) Find the area of the image of R algebraically by computing the Jacobian of the map from uv-space to xy-space.

(c) How do the rows of the Jacobian matrix correspond to the edges of the image of R?

2. What will the Jacobian in the general change of coordinates formula be if

   (a) $x = f(u)$, $y = v$ and $z = w$?

   (b) $x = f(u)$, $y = g(v)$ and $z = h(w)$?

Problems

1. Let E be the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
Use the change of variables $x = au$, $y = bv$ and $z = cw$ to compute the integral $\iiint\limits_E 1\, dV$.

2. Let Q be the quadrilateral in the xy-plane with vertices $(1, 0)$, $(4, 0)$, $(0, 1)$, and $(0, 4)$. Evaluate
$$\iint\limits_Q \frac{1}{x+y}\, dA$$
with the change of variables $x = u - uv$ and $y = uv$.

23. Gravitational Potential Energy

On August 7,1665 Cambridge University was closed down due to an outbreak of bubonic plague. Isaac Newton, then 23 years old, was forced to continue his studies at his mother's country estate in Woolsthorpe. Working in isolation, within a period of two years, he made some of his greatest discoveries in mathematics, optics, and mechanics. However, Newton withheld publication of his gravitational theories for quite some time, until he could prove that a spherical planet has the same gravitational field that it would have if its mass were concentrated at the center. Using multiple integrals and spherical coordinates, we shall solve Newton's problem below.

Problems

1. The gravitational potential energy between two *point* masses m_1, m_2 at a distance d apart is given by the formula $G\frac{m_1 m_2}{d}$, where $G =$ a constant.

   (a) Write down a triple integral that expresses the potential energy between an asteroid in the shape of a ball of radius 1 centered at 0, with uniform mass density ρ_0, and a unit mass at a point on the z-axis, at distance $d \geq 1$ from 0.

   (b) Compute the integral.

2. By symmetry, the calculation above applies to all points at the same distance d from the origin. This defines a function $U(x, y, z)$, the potential energy due to the mass distribution.

   (a) Write down the $U(x, y, z)$, for points outside the ball.

   (b) The gravitational force of the mass distribution $\mathbf{F}$ is a vector quantity given by the formula $\mathbf{F} = -\nabla U$. What is $\mathbf{F}$?

   (c) Show that $\|\mathbf{F}\|$ varies according to the inverse square law, for points outside the ball.

3. Suppose we have a body of uniform mass density. Does the gravitational force outside the body always point towards the center of mass, as in the case of the asteroid? Justify your answer.

Additional Problem

1. What is the formula for $U(x, y, z)$ in the case $d < 1$?

References

http://wwwcn.cern.ch/∼mcnab/n/
http://newton.gws.uky.edu/lifeandthought.html

24. Vector Fields

A *vector field* is just a function $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$. As an example, imagine the waters of the San Francisco bay. The water has a certain velocity at each point in the bay which we could indicate by fixing an arrow on the water. This is the velocity vector field of the bay. If we release a marker of neutral buoyancy (that is, it has no tendency to rise or sink) into the bay, then the path in which it drifts is called a *flow line*. A vector field in the plane and a flow line are seen in Figure 22.

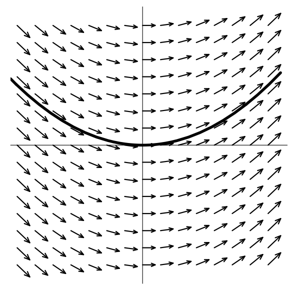


Figure 22: The vector field $\mathbf{F}(x, y) = 1\mathbf{i} + x\mathbf{j}$ and a flow line.

Questions

1. Sketch the vector field $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$.

2. Consider the vector fields

$$\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$$

and

$$\mathbf{G}(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}.$$

(a) Show that $\mathbf{F}(x, y)$ and $\mathbf{G}(x, y)$ have unit length for all (x, y) where they are defined.

(b) Show that $\mathbf{F}(x, y)$ is perpendicular to $\mathbf{G}(x, y)$ for all (x, y) where they are defined.

(c) Sketch the vector fields $\mathbf{F}$ and $\mathbf{G}$.

Problems

1. A *flow line* for a vector field $\mathbf{F}$ is a parametrized path $\mathbf{r}(t)$ such that for every t,

$$\mathbf{r}'(t) = \mathbf{F}(\mathbf{r}(t)).$$

   In words, the velocity vectors along the parametrized path coincide with the vectors of the vector field at all points along the path. Determine whether the following statements are true or false.

   (a) $\mathbf{r}(t) = (t, t^2)$ is a flow line for $\mathbf{F}(x, y) = \mathbf{i} + 2x\mathbf{j}$.

   (b) $\mathbf{r}(t) = (t, t^2 + 5)$ is a flow line for $\mathbf{F}(x, y) = \mathbf{i} + 2x\mathbf{j}$.

   (c) $\mathbf{r}(t) = (\cos t, \sin t)$ is a flow line for $\mathbf{F}(x, y) = \frac{-y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$.

   (d) $\mathbf{r}(t) = (2\cos t, 2\sin t)$ is a flow line for $\mathbf{F}(x, y) = \frac{-y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$.

2. Find some flow line for the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

3. Can two *different* flow lines for a given vector field ever intersect?

25. Line Integrals

Two things that can be integrated over a curve are scalar-valued functions and vector-valued functions. A scalar-valued function $f(x_1, \ldots, x_n)$ can be thought of as a linear density along a curve. Then the integral with respect to arc length gives the total mass of the curve. First we parametrize the curve C by $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ such that $a \leq t \leq b$. Then

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt = \int_a^b f(x_1(t), \ldots, x_n(t))\sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}\, dt.$$

By integrating a vector-valued function $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ over a curve we are finding to what extent the vector field is "pushing" in the direction of the curve. If the vector-valued function represents force, then $\int_C \mathbf{F}\, d\mathbf{r}$ is the work done moving a particle along the curve from one end to the other. If $\mathbf{F}$ represents fluid flow then $\int_C \mathbf{F} \cdot d\mathbf{r}$ measures the fluid flow along the curve C. Unlike the case of the integral of a scalar-valued function, it is now important that the curve have an orientation: you do more work walking up a flight of stairs than walking down a flight of stairs. As above we parametrize the curve C by $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$ such that $a \leq t \leq b$. Then

$$\begin{aligned} \int_c \mathbf{F} \cdot d\mathbf{r} &= \int_C F_1(x_1, \ldots, x_n)\, dx_1 + \cdots + \int_C F_n(x_1, \ldots, x_n)\, dx_n \\ &= \int_a^b F_1(x_1(t), \ldots, x_n(t))\frac{dx_1}{dt}\, dt + \cdots + \int_a^b F_n(x_1(t), \ldots, x_n(t))\frac{dx_n}{dt}\, dt \end{aligned} \qquad (8)$$

Questions

1. Let C be a parametrized curve. Why does $\int_C ds$ give the arc length of C?
2. Can you show, based on Equation 8, that
$$-\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-\mathbf{F}) \cdot d\mathbf{r}?$$
3. Suppose $\mathbf{F}(x, y, z)$ is a vector field and the curves C and C' are the same except they have opposite orientations. What is the relationship between $\int_C \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$?

Problems

1. Let $f(x, y) = x - y$ and $C = \{(x, y) \mid x^2 + y^2 = 1\}$.
   (a) Compute $\int_C f(x, y)\, ds$.
   (b) Show that $f(-x, -y) = -f(x, y)$.

(c) Could the result of (a) be seen directly from the symmetry of f seen in (b)?

2. Let C be the line segment from $(0,0)$ to $(1,1)$. Calculate $\int_C ds$ using the parametrizations $\mathbf{r}_1(t) = (t,t)$, $\mathbf{r}_2(t) = (t^2,t^2)$, and $\mathbf{r}_3(t) = (t^3,t^3)$. We say that the line integral is *independent of parametrization.*

3. Let $\mathbf{F}(x,y) = (y^2+1)\mathbf{i} + (2xy-2)\mathbf{j}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

   (a) C is the straight-line path from $(0,0)$ to $(1,1)$.

   (b) C is the path from $(0,0)$ to $(1,1)$ along the path going straight up, and then straight right.

   (c) C is the path from $(0,0)$ to $(1,1)$ along the parabola $y = x^2$.

26. The Fundamental Theorem of Line Integrals

In this section we meet a generalized version of the fundamental theorem of calculus. Recall from single-variable calculus that $\int_a^b f(x)\,dx = g(b) - g(a)$ where g is a function whose derivative is f. Now suppose that C is a curve of $\mathbf{R}^n$ whose starting and ending points are $\mathbf{a}$ and $\mathbf{b}$. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = g(\mathbf{b}) - g(\mathbf{a})$ where g is a function whose gradient is $\mathbf{F}$. The function g is called a *potential function* for $\mathbf{F}$. A big difference between the single-variable case and this one is that while $f(x)$ always has an antiderivative, $\mathbf{F}(x_1, \ldots, x_n)$ may not have a potential function. If $\mathbf{F}$ does have a potential function, then $\mathbf{F}$ is said to be *conservative.*

We determine whether $\mathbf{F}$ is conservative by using the theorem on equality of mixed partials as follows. Suppose $\mathbf{F} = \nabla g$ for some g; that is, $\langle f_1, \ldots, f_n \rangle = \langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \rangle$. Then by equality of mixed partials,

$$\mathbf{F} \text{ conservative} \Rightarrow \frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial g}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial g}{\partial x_j} = \frac{\partial f_j}{\partial x_i}. \tag{9}$$

So if someone hands us a suspicious $\mathbf{F}(x_1, \ldots, x_n)$ and we find that $\dfrac{\partial f_i}{\partial x_j} \neq \dfrac{\partial f_j}{\partial x_i}$ then we know it isn't conservative. For example, if $\mathbf{F}(x,y) = y\mathbf{i} + 2x\mathbf{j}$ then $\frac{\partial}{\partial y} f_1(x,y) = \frac{\partial}{\partial y} y = 1$ but $\frac{\partial}{\partial x} f_2(x,y) = \frac{\partial}{\partial x} 2x = 2$, so $\mathbf{F}$ has no potential function.

There is a converse to Equation 9 but we need to be careful about the domain of $\mathbf{F}$. On a simply-connected domain, if $\mathbf{F}$ satisfies Equation 9 then $\mathbf{F}$ is conservative. Simply connected regions are regions without holes, as illustrated in Figure 23.

Figure 23: A domain which is simply-connected and a domain which isn't.

If g can exist under the criterion of Equation 9, then we can try to compute it explicitly. Given the partial derivatives $\frac{\partial g}{\partial x_i}$ we find the "anti-partial-derivative" with respect to x_i — bearing in mind that since partial differentiation treats the other variables as constants, our constant of integration could be a function of any of the other variables. Doing this for each x_i gives n formulations of g; setting these equal will eliminate the uncertainty about which other variables occur in the various constants of integration.

Questions

1. Does the fundamental theorem of single-variable calculus follow from the fundamental theorem for line integrals?

2. Suppose **F** is the gradient of some function. What is the work done by **F** along a *closed* curve (i.e., a curve that comes back to where it started)?

3. Suppose two different curves C and C' have the same starting point and ending point.

   (a) If **F** is the gradient of some function, must

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}?$$

   (b) If

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r},$$

   must **F** be the gradient of some function?

4. Explain the results of Problem 3 of Worksheet 24 in terms of the fundamental theorem of line integrals.

5. If $\nabla g_1(x_1, \ldots, x_n) = \nabla g_2(x_1, \ldots, x_n)$, what can you say about the functions g_1 and g_2? (**Hint:** Consider $\nabla(g_1 - g_2)$.)

Problems

1. Which of the following are conservative vector fields? For each conservative vector field, find a function for which it is the gradient.

   (a) $\mathbf{F}(x, y) = (2xy + y)\mathbf{i} + (x^2 + x)\mathbf{j}$.

   (b) $\mathbf{F}(x, y) = \sin(xy)\mathbf{i} + \mathbf{j}$.

   (c) $\mathbf{F}(x, y) = y^2\mathbf{i} + (2xy + x)\mathbf{j}$.

   (d) $\mathbf{F}(x, y, z) = x^2\mathbf{j} + 3xz\mathbf{k}$.

2. Calculate the line integral $\int_c \mathbf{F} \cdot d\mathbf{r}$, where

   $$\mathbf{F}(x, y, z) = (ye^z - 2xy)\mathbf{i} + (xe^z - x^2)\mathbf{j} + (xye^z)\mathbf{k}$$

   and

   $$C = \{(\cos t, \sin t, t) \mid 0 \le t \le 2\pi\}$$

   oriented with increasing t.
   (**Hint:** Can you do this without actually evaluating the line integral?)

3. Consider

   $$\mathbf{F}(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}.$$

   (a) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle oriented counter-clockwise by parametrizing C and directly integrating.

(b) Show that **F** satisfies Equation 9.

(c) Let D be the portion of $\mathbf{R}^2$ where **F** is defined. Is D simply connected?

(d) Do (a) and (b) together contradict your answer to question 2?

27. Green's Theorem

Green's theorem gives a special technique for evaluating line integrals in the plane when the domain of integration is a closed curve (i.e., a curve whose starting point is the same as its ending point). Let $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field. The theorem says that if C is a closed curve in $\mathbf{R}^2$ and D is the region of the plane that C encloses, and if the orientation of C is counter-clockwise around D, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y)\, dx + Q(x, y)\, dy = \iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dA.$$

If we regard $\mathbf{F}$ as the velocity vector field of a fluid flowing in the plane then we can interpret the expression $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}$ as follows: at each point it measures the tendency of a paddle wheel suspended in the fluid to rotate counter-clockwise. For this reason, the scalar-valued function $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}$ is called the *circulation*.

Questions

1. Suppose C is a closed curve in the plane oriented *clockwise*. Does Green's theorem give us any information about $\int_C \mathbf{F} \cdot d\mathbf{r}$?

2. Let C be a closed curve in the plane and let $\mathbf{F}$ be conservative.

   (a) Use the fundamental theorem of line integrals to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   (b) Use Green's theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

3. It appears as if Green's theorem tells us that

$$\int_C x\, dx = \iint_D 0\, dx\, dy = 0.$$

   But we know from single-variable calculus that

$$\int x\, dx = \frac{x^2}{2} + C.$$

   Is something amiss?

4. Let C be a closed curve. What geometric quantity is computed by

$$\frac{1}{2} \int_C -y\, dx + x\, dy?$$

   There is a device used by surveyors called a *mechanical integrator* that uses this fact to find areas by tracing out boundaries.

Problems

1. Compute $\int_C y^2\,dx + x\,dy$ where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ oriented counter-clockwise.

2. Let $D = \{(x, y) \mid a \le x \le b, c \le y \le d\}$ be a rectangle, C be its boundary oriented counter-clockwise, and $\mathbf{F}(x, y) = P(x, y)\mathbf{i}$.

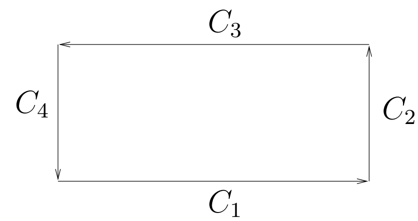


Figure 24: An oriented rectangle.

   (a) Explain in terms of work why

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \cdots + \int_{C_4} \mathbf{F}\cdot d\mathbf{r},$$

   where the C_i are the edges of the rectangle as in Figure 24.

   (b) Show that

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_4} \mathbf{F}\cdot d\mathbf{r} = 0$$

   by parametrizing C_2 and C_4 and directly integrating.

   (c) Show that

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_a^b P(x, c) - P(x, d)\,dx$$

   by parametrizing C_1 and C_3 and directly integrating.

   (d) Let x be a fixed constant. Compute

$$\int_c^d -\frac{\partial P}{\partial y}(x, y)\,dy$$

   using the single-variable fundamental theorem.

   (e) Conclude that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D -\frac{\partial P}{\partial y}\,dx\,dy.$$

3. How is problem 2 related to Green's theorem?

28. Curl and Divergence

In this section we define two operations on vector fields and discuss their physical significance.

Algebraically, if $\mathbf{F}(x, y, z)$ is a vector field given by

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k},$$

then we define

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}.$$

Note that the curl is a vector-valued function defined only for vector fields in three-space. If we think of $\mathbf{F}(x, y, z)$ as the velocity vector field of a fluid in motion, then we can interpret the curl physically. Its direction tells us how to suspend a paddle wheel in a fluid so that it spins most rapidly. Specifically, at the point (x, y, z) we align the axis of the paddle wheel along the vector $(\text{curl}\,\mathbf{F})(x, y, z)$. Then if we view the paddle wheel as in Figure 25 it will appear to be rotating in the counter-clockwise direction. The magnitude of curl $\mathbf{F}$ tells us how rapidly it will rotate. For this reason, when curl $\mathbf{F} = 0$ we say that $\mathbf{F}$ is *irrotational*.

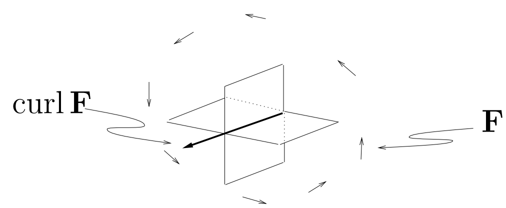


Figure 25: A paddle wheel suspended in a moving fluid.

Algebraically, the divergence is defined for any vector field in $\mathbf{R}^n$ as

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x_1}F_1 + \cdots + \frac{\partial}{\partial x_n}F_n = \nabla \cdot \mathbf{F}.$$

Note that div $\mathbf{F}$ is a scalar-valued function. Physically, the divergence tells us the extent to which each point in space is a source of fluid (positive divergence) or a sink of fluid (negative divergence). For this reason, when div $\mathbf{F} = 0$ we say the fluid is *incompressible*.

There are two useful facts which one may prove by direct computation:

The curl of the gradient of any function is zero. $(\nabla \times \nabla\phi = 0)$

The divergence of the curl of any vector field is zero. $(\nabla \cdot \nabla \times \mathbf{F} = 0)$

Questions

1. Let $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 0\mathbf{k}$.

   (a) Sketch $\mathbf{F}$ in the xy-plane.

(b) Compute curl $\mathbf{F}$ and include it in your sketch from part (a).

(c) What is curl $\mathbf{F}$ telling us about the fluid flow?

2. Let $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$, and $\mathbf{G}(x, y) = -x\mathbf{i} - y\mathbf{j}$.

   (a) Sketch $\mathbf{F}$ and $\mathbf{G}$.

   (b) Compute div $\mathbf{F}$ and div $\mathbf{G}$.

   (c) For both $\mathbf{F}$ and $\mathbf{G}$, state if the origin is a fluid source or sink.

3. Let Figure 26 be a two-dimensional planar cross section of a three-dimensional vector field $\mathbf{F}(x, y, z)$.

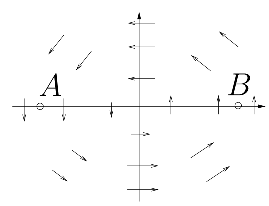


Figure 26: A whirlpool.

   (a) Is curl $\mathbf{F}$ zero at the origin? Why?

   (b) Suppose we wish to swim from A to B. Draw a hard path C, and an easy path C', which we could take.

   (c) Is $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$?

   (d) Can $\mathbf{F}$ be the gradient of some function $g(x, y, z)$?

   (e) Let $g(x, y, z)$ be a scalar-valued function. Based on parts (a)–(d), explain why curl ∇g must be zero.

Problems

1. Let $\mathbf{F}(x, y, z) = (1 + ze^y)\mathbf{i} + xze^y\mathbf{j} + xe^y\mathbf{k}$.

   (a) Can $\mathbf{F}$ be the gradient of some function $f(x, y, z)$?

   (b) If the answer to part (a) is "yes," find all functions $f(x, y, z)$ of which $\mathbf{F}$ is the gradient.

2. Let $\mathbf{F}(x, y, z) = -\mathbf{i} + (y - e^x)\mathbf{j} - z\mathbf{k}$.

   (a) Can $\mathbf{F}$ be the curl of some vector field $\mathbf{G}(x, y, z)$?

   (b) If the answer to part (a) was "yes," find all functions $\mathbf{G}(x, y, z)$ of which $\mathbf{F}$ is the curl.

3. Suppose that $\mathbf{F}$ and $\mathbf{G}$ are vector fields in $\mathbf{R}^3$, and define the vector field

$$\mathbf{H}(x, y, z) = \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z).$$

Prove that

$$\operatorname{div} \mathbf{H} = \mathbf{G} \cdot (\operatorname{curl} \mathbf{F}) - \mathbf{F} \cdot (\operatorname{curl} \mathbf{G}).$$

4. Suppose that $f(x, y, z)$ and $g(x, y, z)$ are real-valued functions and define the vector field

$$\mathbf{H}(x, y, z) = \nabla f(x, y, z) \times \nabla g(x, y, z).$$

Prove that $\operatorname{div} \mathbf{H} = 0$.

29. Parametric Surfaces and Their Areas

Parametric descriptions of surfaces have similar advantages to parametric descriptions of curves. For example, not all surfaces form the graph of some function of two variables. We parametrize a surface by means of a function whose domain is some subset of $\mathbf{R}^2$, called the *parameter domain*, and whose range is in $\mathbf{R}^3$. For example, a map of the world is a portion of the plane, and there is an obvious function from this portion of the plane into the three-dimensional universe. In the Mercator projection the two parameters are longitude and latitude.

Suppose the parameters are u and v and $x(u,v)$, $y(u,v)$ and $z(u,v)$ are the coordinates in $\mathbf{R}^3$ corresponding to (u,v). Consider a small rectangle A in the parameter domain of width Δu and height Δv. We can find an approximate image for the lower edge of A by using partial derivatives to estimate how x, y, and z change as u changes by Δu and v is constant:

$$\Delta x \approx \frac{\partial x}{\partial u}\Delta u, \qquad \Delta y \approx \frac{\partial y}{\partial u}\Delta u, \qquad \Delta z \approx \frac{\partial z}{\partial u}\Delta u.$$

Thus, the image of the vector $\Delta u\,\mathbf{i}$ is the vector $\left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right)\Delta u$. Similarly, the image of left hand edge (the vector $\Delta v\,\mathbf{j}$) is $\left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right)\Delta v$. These two vectors span a parallelogram tangent to the surface which approximates the image of A in xyz-space as in Figure 27. We calculate its area by taking the cross-product:

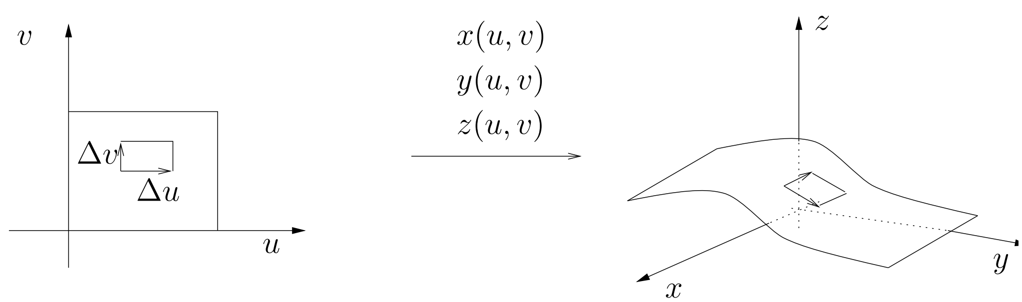


Figure 27: Tangent vectors to a parametric surface.

$$\begin{aligned}\text{area of image of } A &\approx \left|\left[\left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right)\Delta u\right] \times \left[\left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right)\Delta v\right]\right| \\ &= \left|\left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right) \times \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right)\right| \Delta u\,\Delta v\end{aligned}$$

Note that the result of the cross-product is a vector perpendicular to the surface.

We may find an estimate for the surface area of the parametric surface by partitioning the parameter domain into small rectangles A_{ij} and summing the areas of their approximate images:

$$\text{surface area} \approx \sum_{i,j} \left|\left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right) \times \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right)\right| \Delta u_i\,\Delta v_j.$$

Passing to the limit of an arbitrarily fine partition gives us the formula for surface area:

$$\text{surface area} = \iint_D \left| \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \right) \times \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} \right) \right| dA, \qquad (10)$$

where D is the entire parameter domain in uv-space.

Questions

1. Suppose $f(u, v)$ is a real-valued function. Then the graph of f is given parametrically as $x = u$, $y = v$, $z = f(u, v)$.

   (a) Use Equation 10 to derive a formula for the surface area of the graph of f over a region $D \subset \mathbf{R}^2$.

   (b) How does your formula compare with Equation 7 on Worksheet 20?

2. Let $x(u, v)$, $y(u, v)$ and $z(u, v)$ be coordinate functions parametrizing a metal plate suspended in space, and let $D \subset \mathbf{R}^2$ be the parameter domain. Suppose $\rho(x, y, z)$ is a density function on the metal plate. Write a formula for the mass of the metal plate.

Problems

1. A sphere of radius a has a polar ice cap extending ϕ_0 radians of latitude from the north pole. Find the area of the ice cap.

2. Consider the cone $C = \{(x, y, z) \mid z = c\sqrt{x^2 + y^2} \text{ where } x^2 + y^2 \leq a\}$ when a and c are constants.

   (a) Parametrize the cone in spherical coordinates.

   (b) Compute the surface area of the cone.

   (c) One can cut a cone and lay it flat with no rips or wrinkles, as in Figure 28. Use this fact to compute the surface area of the cone.

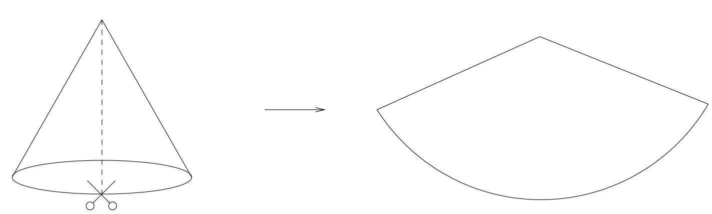


Figure 28: Cut a cone and lay it flat!

3. Let D be the domain $\{(u,v) \mid 0 \le u \le 2\pi, -1 \le v \le 1\}$. Consider the parametric surface described by the equations $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ where

$$\begin{aligned} x(u,v) &= 2\cos u + v\sin\frac{u}{2}\cos u \\ y(u,v) &= 2\sin u + v\sin\frac{u}{2}\sin u \\ z(u,v) &= v\cos\frac{u}{2} \end{aligned}$$

(a) Consider the vector $\mathbf{r}(u,v)$ as a sum of two vectors : $\mathbf{r}(u,v) = \mathbf{C}(u) + \mathbf{F}(u,v)$ where $\mathbf{C}(u) = (2\cos u, 2\sin u, 0)$ and $\mathbf{F}(u,v) = v(\sin\frac{u}{2}\cos u, \sin\frac{u}{2}\sin u, \cos\frac{u}{2})$

i. As u varies from 0 to 2π, what curve does $\mathbf{C}(u)$ sweep out in R^3?

ii. What is $\|\mathbf{F}\|$?

iii. What angle does $\mathbf{F}$ make with the z-axis? Hint: Compute $\mathbf{F} \cdot \mathbf{k}$, where $\mathbf{k} = (0,0,1)$.

iv. Check that the projection of $\mathbf{F}(u,v)$ into the xy-plane is a multiple of $\mathbf{C}(u)$.

v. For the values $v = \{-1, 1\}$, $u = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$, draw the vector $\mathbf{F}(u,v)$, with tail at $\mathbf{C}(u)$. The tip of $\mathbf{F}(u,v)$ then points to $\mathbf{r}(u,v)$.

(b) Now allow to v vary between -1 and 1, and u between 0 and 2π. What surface is described by $\mathbf{r}(u,v)$?

(c) Compute the normal $\mathbf{r}_u \times \mathbf{r}_v$ to the surface along the curve C for the values $u = 0$ and $u = 2\pi$. Intuitively, your result explains why this surface in not "orientable" or "has only one side". This will be discussed later in the course.

30. Surface Integrals

In this section we continue our investigation of integrals over surfaces. Suppose S is a surface suspended in a moving fluid, i.e., a permeable membrane. What is the rate of fluid flow across the surface? If the surface is flat and the velocity is the same at all points on the surface, then the question is relatively simple to answer.

1. Find the rate of fluid flow in the direction normal to the surface.

2. Multiply by the area of the surface.

Let $\mathbf{F}$ be the velocity of the fluid and let $\mathbf{n}$ be a *normal* vector (this means that the vector is perpendicular to the surface and of unit length). Then the magnitude of the component

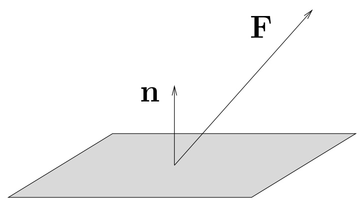


Figure 29: A surface with fluid flowing across it.

of $\mathbf{F}$ perpendicular to the surface is $\mathbf{F}\cdot\mathbf{n}$. If we denote the area of the surface by ΔS then the rate of fluid flow across the surface is $(\mathbf{F}\cdot\mathbf{n})\,\Delta S$.

Now suppose that the surface is curved and that $\mathbf{F}(x,y,z)$ can vary. To find the rate of fluid flow we divide the surface into small enough regions so that $\mathbf{F}(x,y,z)$ is roughly constant on each region, and then proceed as above. First we need to find a normal vector to the surface. Recall from the previous section that if the surface is parametrized in terms of (u,v), then a perpendicular vector to the surface is given by

$$\left(\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k}\right)\times\left(\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k}\right).$$

Dividing this vector by its length we get a unit normal vector to the surface:

$$\mathbf{n}=\frac{(\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k})\times(\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k})}{\left|(\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k})\times(\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k})\right|}.$$

Recall also that we know how to find the area of a little parallelogram tangent to the surface:

$$\Delta S=\left|\left(\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k}\right)\times\left(\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k}\right)\right|\,\Delta u\,\Delta v.$$

Thus, the rate of fluid flow across the little parallelogram is approximately

$$(\mathbf{F}\cdot\mathbf{n})\,\Delta S=\mathbf{F}\cdot\left[\left(\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k}\right)\times\left(\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k}\right)\right]\,\Delta u\,\Delta v.$$

Summing over all the small parallelograms and passing to the limit of an arbitrarily fine partition of the parameter domain gives the formula

$$\begin{aligned}\text{rate of fluid flow across } S &= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS\\ &= \iint_D \mathbf{F}\cdot\left[\left(\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k}\right)\times\left(\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k}\right)\right]\,dA,\end{aligned}$$

where D is the parameter domain.

The flow of a vector field through a surface is called the *flux across* the surface.

Questions

1. On the surface parametrized by
$$x = u, \quad y = v, \quad z = f(u, v),$$
sketch the vectors
$$\frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k},$$
$$\frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k},$$
and their cross-product.

2. On the surface parametrized by
$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi,$$
sketch the vectors
$$\frac{\partial x}{\partial \phi}(\phi_0, \theta_0)\mathbf{i} + \frac{\partial y}{\partial \phi}(\phi_0, \theta_0)\mathbf{j} + \frac{\partial z}{\partial \phi}(\phi_0, \theta_0)\mathbf{k},$$
$$\frac{\partial x}{\partial \theta}(\phi_0, \theta_0)\mathbf{i} + \frac{\partial y}{\partial \theta}(\phi_0, \theta_0)\mathbf{j} + \frac{\partial z}{\partial \theta}(\phi_0, \theta_0)\mathbf{k},$$
and their cross-product.

Problems

1. Let
$$S = \{(x, y, z) \mid z^2 = x^2 + y^2,\ 0 \le z \le 1\}.$$
Orient S with a normal having positive $\mathbf{k}$-component. Compute
$$\iint_S \mathbf{F}\cdot d\mathbf{S},$$
where

(a) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(b) $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$.

2. (a) Compute the flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the sphere $x^2 + y^2 + z^2 = a^2$ where a is a constant.

   (b) What is the magnitude of $\mathbf{F}(x, y, z)$ at every point on the sphere?

   (c) What is the surface area of the sphere?

   (d) Explain how to combine (b) and (c) to get your answer to part (a). Why does this work?

3. Suppose S is a surface of finite area and $\mathbf{F}(x, y, z) = \mathbf{C}$ is constant on S. Is the flux of $\mathbf{F}$ across S just $|\mathbf{C}|$ times the area of S?

31. Stokes' Theorem

Stokes' theorem is the analogue of Green's theorem for a surface in $\mathbf{R}^3$. To be precise, let S be an oriented surface in $\mathbf{R}^3$ and let C be a curve forming the boundary of S with orientation compatible with that of S, as in Figure 30. Let $\mathbf{F}(x, y, z)$ be a vector field in $\mathbf{R}^3$.

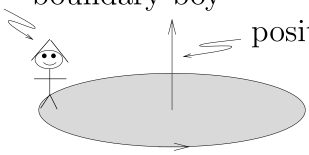


Figure 30: An oriented surface with a compatibly oriented boundary

Then Stokes' theorem states that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Some surfaces, such as the surface of a sphere, have no boundary. Stoke's theorem still works in these cases, but notice that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

What does it mean for the orientation of the boundary of C to be "compatible" with the orientation of the surface S? Well, shrink yourself down and stand on the same side of the surface as the positive normal, as in Figure 30. Now walk around the boundary so that the surface is on your left-hand side. The way you are walking is in the direction of the "compatible" orientation for the boundary.

Questions

1. Let $\mathbf{F}(x, y, z)$ be a vector field.

   (a) What does Stoke's theorem imply about the flux of $\operatorname{curl} \mathbf{F}$ through the upper unit hemisphere $x^2 + y^2 + z^2 = 1$ where $z \geq 0$?

   (b) What does Stoke's theorem imply about the flux of $\operatorname{curl} \mathbf{F}$ through the lower unit hemisphere $x^2 + y^2 + z^2 = 1$ where $z \leq 0$?

   (c) What is the flux of $\operatorname{curl} \mathbf{F}$ through the unit sphere $x^2 + y^2 + z^2 = 1$? (**Hint:** Use parts (a) and (b).)

   (d) What can you say about the flux of $\mathbf{F}$ through unit sphere?

2. Show that Stokes' theorem implies Green's theorem as follows. Let

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

be a vector field in the plane, and define a vector field on $\mathbf{R}^3$ by

$$\mathbf{G}(x, y, z) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} + 0\mathbf{k}.$$

Suppose the surface S is a region in the xy-plane and the curve C is the boundary of S.

(a) Show that

$$\int_C P(x, y)\,dx + Q(x, y)\,dy = \int_C \mathbf{G} \cdot d\mathbf{r}.$$

(b) Show from Stokes' theorem that

$$\int_C P(x, y)\,dx + Q(x, y)\,dy = \iint_S \operatorname{curl} \mathbf{G} \cdot d\mathbf{S}.$$

(c) Show that the equation of part (b) is exactly Green's theorem by explicitly writing out the right-hand side.

Problems

1. Let S be the the portion of the surface $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$, oriented with positive normal pointing away from the origin. Let

$$\mathbf{F}(x, y, z) = (y - z)\mathbf{i} - (x + z)\mathbf{j} + (x + y)\mathbf{k}.$$

Compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

2. Let C be the triangle in $\mathbf{R}^3$ with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 1)$. Compute

$$\int_C (x^2 + y)\,dx + yz\,dy + (x - z^2)\,dz.$$

3. Let S be the surface parametrized by

$$x = uv, \quad y = u + v, \quad z = u^2 + v^2$$
$$u \leq 1, \quad v \geq 0, \quad \text{and } v \leq u.$$

Let $\mathbf{F}(x, y, z) = x\mathbf{k}$ and compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS.$$

Additional Problems

1. Can you think of a way to apply Stokes' theorem to find the area of a surface in $\mathbf{R}^3$ by constructing some vector field $\mathbf{F}(x, y, z)$ and taking its line integral over the boundary of the surface?

32. The Divergence Theorem

The divergence theorem is our last generalization of the fundamental theorem of calculus. Let E be a solid region of $\mathbf{R}^3$ and let the surface S form the boundary of E. Orient S with positive normal pointing out of E. Then if $\mathbf{F}(x, y, z)$ is a vector field, the divergence theorem states that

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_E \operatorname{div} \mathbf{F}\, dV.$$

Questions

1. Suppose that $\mathbf{F}(x, y, z)$ is an incompressible vector field and that S is the boundary of a solid E. What is $\iint\limits_S \mathbf{F} \cdot d\mathbf{S}$?

2. Let E be a solid with boundary S. What geometric quantity is computed by
$$\frac{1}{3} \iint\limits_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{S}?$$

3. Let S be a sphere and let $\mathbf{F}(x, y, z)$ be a vector field. Use the divergence theorem to compute $\iint\limits_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

4. Write down a one-dimensional version of the divergence theorem. What is the name of this famous theorem?

Problems

1. Let $f(x, y, z) = x^2 + y^2 + z^2$ and let $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$. Compute the flux of $\mathbf{F}$ outward through the sphere $x^2 + y^2 + z^2 = 1$.

2. Let $\mathbf{F}(x, y, z)$ be a vector field. Suppose S_1 and S_2 are two oriented surfaces in $\mathbf{R}^3$ that have the same oriented boundary C, but don't intersect anywhere else. Therefore, the union of S_1 and S_2 is a surface bounding a solid region E in $\mathbf{R}^3$. Show that
$$\int\limits_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int\limits_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$
   (a) using Stokes' theorem
   (b) using the divergence theorem

Additional Problem

1. Can you formulate a two-dimensional divergence theorem?