Final Exam

Introduction to the Theory of Sets

Math 135, Fall 2008

Problem 1 (30 points): Carefully read each of the following statements, and decide whether it is true or false. Circle your answer accordingly.

(1.1) (True) (False) The order $(P(\mathbb{N}), \subseteq)$ is a preordering, but not a poset, because it is not antisymmetric.

(1.2) (True) (False) Zorn's Lemma is the statement that every non-empty poset has a maximal chain.

(1.3) (True) (False) Without assuming the Axiom of Choice, we cannot conclude that the map which sends a set x to $\{x, \{x\}\}$ is a definite operation.

(1.4) (True) (False) The set $\bigcup\{\{\mathbb{N}\}\}$ is finite.

(1.5) (True) (False) If A is transitive, then $TR(A)$, the transitive closure of A, is equal to $A \cup \{A\}$.

(1.6) (True) (False) There exist classes which are sets, but some classes are not sets because they are too large.

(1.7) (True) (False) The Continuum Hypothesis is a consequence of the Countable Axiom of Choice.

(1.8) (True) (False) The Axiom of Extensionality implies there are no atoms, because if x is an atom, it has nothing in it, and therefore by the Axiom of Extensionality it is equal to the empty set.

(1.9) (True) (False) Historically, it was fairly easy to find a foundation of mathematics, because the axioms of set theory are so obvious.

(1.10) (True) (False) The transitive closure of the set $\{0, 1, \{\mathbb{N}\}\}$ is uncountable, because it contains every subset of $\mathbb{N}$.

(1.11) (True) / (False) Let $(\alpha, \leq)$ be an ordinal with the usual ordering. Then for all x in α, $x = \mathrm{seg}_{(\alpha,\leq)}(x)$.

(1.12) (True) / (False) A relation $(U, <)$ is extensional if whenever x and y are in U and for all z in U, $z < x$ iff $z < y$, then $x = y$.

(1.13) (True) / (False) A set A is transitive iff $x \subseteq A$ implies $x \in A$.

(1.14) (True) / (False) If A is a transitive set and for any distinct x and y in A, $x \in y$ or $y \in x$, then A is an ordinal.

(1.15) (True) / (False) The empty set is not an ordinal, but the power set of the empty set is an ordinal.

(1.16) (True) / (False) The class of all ordinals is itself an ordinal.

(1.17) (True) / (False) There are sets of ordinals which are not themselves ordinals, but every transitive set of ordinals is an ordinal.

(1.18) (True) / (False) Every ordinal α has a successor which is $\alpha \cup \{\alpha\}$.

(1.19) (True) / (False) Every set of ordinals has a least element, but there exist classes of ordinals which do not have least elements because they are too large.

(1.20) (True) / (False) Every ordinal is either equal to 0, is a successor ordinal, or is a limit ordinal; there are no other possibilities.

(1.21) (True) / (False) If α is an ordinal and $\pi : \alpha \to \text{o.t.}(\alpha)$ is the Mostowski collapse map, then for all x in α, $\pi(x) = x$.

(1.22) (True) / (False) There are exactly ω many countable ordinals.

(1.23) (True) / (False) If $\alpha < \kappa$ and κ is a cardinal, then there is a bijection of κ onto α.

(1.24) (True) / (False) If A is a set of ordinals, then $\bigcup A$ is an ordinal.

(1.25) (True) / (False) For any ordinals $\alpha < \beta$ and γ, $\alpha + \gamma < \beta + \gamma$.

(1.26) (True) / (False) $(\omega + \omega) \cdot \omega < (\omega \cdot \omega) + 1$.

(1.27) (True) / (False) Every regular ordinal is a cardinal.

(1.28) (True) / (False) Every singular ordinal is not a cardinal.

(1.29) (True) / (False) There exists an unbounded subset of ω_5 which has size ω_4.

(1.30) (True) / (False) If $\omega_2 < \alpha$ then α has uncountable cofinality.

Problem 2 (5 points):

State the Schröder-Bernstein Theorem. State the Axiom of Choice. State the theorem on the Comparability of Well-Orders. Describe Russell's Paradox.

Problem 3 (10 points):

Using the Axiom of Choice, prove that if there is a surjection $g : B \to A$, then there is an injection $f : A \to B$. (Be precise in your application of the Axiom of Choice.)

Problem 4 (10 points):

Define what it means for a set A to be transitive. Is the set $\{\emptyset, \{\emptyset\}\}$ transitive? Let A be a set and define by recursion a sequence $\langle A_n : n < \omega \rangle$ by letting $A_0 = A$ and $A_{n+1} = \bigcup A_n$. Prove that $B = \bigcup\{A_n : n < \omega\}$ is transitive.

Problem 5 (5 points):

Carefully describe what it means to define a function (on a well-order or on a well-ordered class) by transfinite recursion. Explain the difference between the transfinite recursion theorem introduced early in the semester and the transfinite recursion theorem which requires the Axiom of Replacement. Define by recursion the definite operations $\alpha \mapsto V_\alpha$ and $\alpha \mapsto \omega_\alpha$.

Problem 6 (10 points):

(a) State the Axioms of Separation, Replacement, Power Set, and Foundation.

(b) Give a specific example of ordinals α, β, and γ such that $(\beta + \gamma) \cdot \alpha$ is not equal to $(\beta \cdot \alpha) + (\gamma \cdot \alpha)$.

(c) Define what is the cofinality of a limit ordinal. Define what is a regular limit ordinal and a singular limit ordinal.

(d) Define the cardinal arithmetic operations $\kappa + \lambda$, $\kappa \cdot \lambda$, and κ^λ.

(e) Define an order (X, R) by letting $X = \{a, b, c\}$ and aRc and bRc. Compute the Mostowski collapse map of (X, R).

Problem 7 (10 points):

Let A be a set of ordinals and let $\pi : A \to \text{o.t.}(A)$ be the Mostowski collapse map. Prove that for all γ in A, $\pi(\gamma) \leq \gamma$. Use this fact to prove that if $A \subseteq \alpha$ then $\text{o.t.}(A) \leq \alpha$. (You may use the fact that for ordinals γ and δ, $\gamma \leq \delta$ iff $\gamma \subseteq \delta$.)

Problem 8 (10 points):

Prove that if there is a strictly monotone function $f : \beta \rightarrow \alpha$, where α and β are ordinals, then $\beta \leq \alpha$. (Hint: Consider $f[\beta]$, which is a subset of α. Now apply Problem 8.)

Problem 9 (10 points):

Define what is a cardinal number. Prove that if A is a set of cardinal numbers, then $\sup(A)$ is a cardinal number.