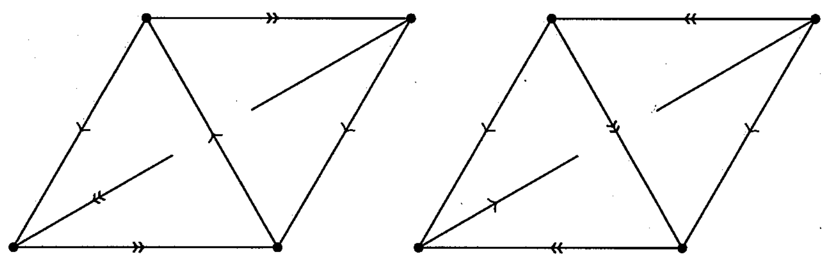
Final Exam - Algebraic Topology

Christian Zickert

Let K be the topological space obtained by gluing together the faces of the two 3–simplices below, using the unique orientation preserving linear identification maps respecting the arrows.



(1) Show that there is no vertex ordering making the identification maps order preserving. Conclude that the gluing does not produce a Δ-complex structure on K.

(2) Show that K has a CW-complex structure with 1 zero-cell, 2 one-cells, 4 two-cells and 2 three-cells.

(3) Show that the fundamental group of K is $\mathbb{Z}/5\mathbb{Z}$.

(4) Write down the cellular chain complex and show that the homology groups of K are given by

$$H_i(K) = \begin{cases} \mathbb{Z} \text{ for } i = 0 \\ \mathbb{Z}/5\mathbb{Z} \text{ for } i = 1 \\ \mathbb{Z} \text{ for } i = 2, 3. \end{cases}$$

(5) Compute the cohomology groups $H^*(K)$ and $H^*(K; \mathbb{Z}/p\mathbb{Z})$ for each prime number p.

(6) Let x_0 denote the zero-cell. Show that x_0 has a neighborhood homeomorphic to the cone on a torus.

(7) Show that $M = K \backslash \{x_0\}$ is a 3–manifold and that M is homotopy equivalent to a 3–manifold $\bar{M}$ with boundary a torus.

(8) Show that $H_3(\bar{M}, \partial\bar{M}) = \mathbb{Z}$ and use this to prove that M and $\bar{M}$ are orientable.

(9) Compute $H_*(M)$ and $H^*(M)$.

(10) For each natural number n, determine the number of connected, abelian n-sheeted covers of M.