Midterm II/math110/fall 2003 LIU
time: Oct 14, 2:10-3:30pm
Total Score: 100%

1. (15%) Determine if each of the following statements is true or false. Give a brief explanation for each of your assertions.

(a). Let V and W be two finite dimensional vector spaces over $\mathbf{R}$, then $\mathcal{L}(V,W)$ and $\mathcal{L}(W,V)$ are isomorphic. (4%)

(b). The null space of a linear transformation $T: V \mapsto V$, $N(T)$ is always contained in the range of T, $R(T)$, i.e. $N(T) \subset R(T)$. (4%)

(c). There can be no 'onto' linear transformation from $\mathbf{R}^{10}$ to $P_{10}(\mathbf{R})$. (3%)

(d). All the $n-$linear functions $\delta : \mathbf{M}_{n\times n}(\mathbf{R}) \mapsto \mathbf{R}$ are equal to the determinant function $det : \mathbf{M}_{n\times n}(\mathbf{R}) \mapsto \mathbf{R}$. (4%)

2. (15%) Let $T: P_3(\mathbf{R}) \mapsto \mathbf{R}^4$ be defined by $T(f) = (f(0), f(1), f'(0), f'(1))$. Let β, γ be the standard ordered bases of $P_3(\mathbf{R})$ and $\mathbf{R}^4$.
(i). Calculate $[T]_\beta^\gamma$. (7%)

(ii). Is T an isomorphism? If yes, prove it. If not, explain to us why not. (8%)

3. (18%)
(i). Prove that a 1-1 linear transformation $T \in \mathcal{L}(V,W)$, $dim(V) = dim(W) < \infty$, must be an isomorphism. (9%)

(ii). Let V be a finite dimensional vector space and let $T \in \mathcal{L}(V)$. Suppose that for some ordered basis β, $det([T]_\beta) = 0$,
(a). prove that $det([T]_\gamma) = 0$ for an arbitrary ordered basis γ of V. (5%)

(b). prove that $null(T) > 0$. (4%)

4. (12%) Let $V \subset P_2(\mathbf{R})$ be the subspace $\{f | f \in P_2(\mathbf{R}), f(1) = 0\}$.
Let $\beta = \{\mathbf{v}_1 = x - 1, \mathbf{v}_2 = x^2 - x\}$ be an ordered basis of V.
(i). Let $\gamma = \{-1\mathbf{v}_1 + a\mathbf{v}_2, b\mathbf{v}_1 + 2\mathbf{v}_2\}$, $a, b \in \mathbf{R}$, be another ordered basis. Calculate the change of coordinate matrix from β to γ by its definition. (4%)

(ii). Let $a = -2, b = -1$ in (i). Suppose that a linear transformation $T: V \mapsto V$ satisfies $[T]_\gamma = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Please determine $[T]_\beta$. (8%)

5. (25%)
(a). State and prove the dimension theorem for a linear transformation $T \in \mathcal{L}(V,W)$, $dimV < \infty$. (15%)

(b). Let $V = \mathbf{M}_{2\times 2}(\mathbf{R})$. Let $\beta = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$ be an ordered basis of V. Determine the dual basis β^* of V^*. (10%)

6. (15%) Let V, W, U be three finite dimensional vector spaces and let α, β, γ be ordered bases for V, W and U, respectively. Prove that for all $S \in \mathcal{L}(W,U)$ and $T \in \mathcal{L}(V,W)$,

$$[ST]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta.$$

Extra Credit: (10%) Let D an $n \times n$ diagonal matrix whose (i,i) entry is equal to i, $1 \le i \le n$.
Define a linear transformation $T: \mathbf{M}_{n\times n}(\mathbf{R}) \mapsto \mathbf{M}_{n\times n}(\mathbf{R})$ by $T(A) = DA - AD$. Find a basis of $N(T)$ and determine $rank(T)$.