MATH 222A – MIDTERM F'03 EVANS

INSTRUCTIONS: This test is due in class on Thursday, October 30. You may refer to the textbook and your own class notes, but do not consult any other book or article or any person. Please come by my office, 907 Evans Hall, if you have any questions about these problems.

Problem #1. Suppose u solves the *Klein–Gordon equation*

$$u_{tt} - \Delta u + m^2 u = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and define the cone $C := \{(x,t) \mid 0 \le t \le t_0, |x - x_0| \le t_0 - t\}$.

Prove that if $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ within the cone C.

Problem #2. Suppose U is a bounded open set, and u is the solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Consider a point $x^0 \in \partial U$, and suppose that the ball $B(0,r)$ lies in $\mathbb{R}^n - U$, with $x^0 \in \partial B(0,r)$. We then say "x^0 is touched from the exterior by the ball $B(0,r)$."

Show that there exists a constant C, depending upon on $n, r, ||f||_{L^\infty}$, but not u, such that

$$\left| \frac{\partial u(x^0)}{\partial \nu} \right| \le C.$$

(Hints: Draw a picture. Define the "barrier" function

$$v(x) = \mu \left(e^{-\lambda |x|^2} - e^{-\lambda r^2} \right).$$

Adjust the constants μ, λ so that $-\Delta v \le -||f||_{L^\infty}$ in U, $v \le 0$ on ∂U. Write $w := v \pm u$. Now use the maximum principle to show that $w \le 0$ in U. Since $w(x^0) = 0$, deduce that $\frac{\partial w}{\partial \nu} \ge 0$ at x^0.)

Problem #3. Let u be a bounded, nonnegative solution of the *heat equation*

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

Fix a radius $r > 0$ and two times $0 < t_1 < t_2$. Prove that there exists a constant C, depending upon n, r, t_1, t_2, but not u, such that

$$\max_{B(0,r)} u(\cdot, t_1) \le C \min_{B(0,r)} u(\cdot, t_2).$$

(Hint: Represent u in terms of its initial data $g = u(\cdot, 0) \ge 0$. Show that there exists a constant C so that $\Phi(x_1 - y, t_1) \le C\Phi(x_2 - y, t_2)$ for all $y \in \mathbb{R}^n$ and all $x_1, x_2 \in B(0,r)$.)