Midterm I/math110/fall 2003 LIU
Total Score: 100%, Time:2:10-3:30pm
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Name:

1.(15%) Determine if the following statements are true or false and give a brief reasoning for each statement.

(a). Every vector space V over $\mathbf{R}$ is finite dimensional. (3%)

(b). The union of any two vector spaces over $\mathbf{R}$ is always a vector space over $\mathbf{R}$. (4%)

(c). The set $V = \{(x,y)|x,y \in \mathbf{R}\}$ with the addition $(x_1,y_1)+(x_2,y_2) = (y_1+x_2, x_1+y_2)$ and scalar multiplication $c\cdot(x,y) = (cx, cy)$, $c \in \mathbf{R}$ forms a vector space over $\mathbf{R}$. (4%)

(d). Any basis of an n dimensional vector space over $\mathbf{R}$ can not contain more than n elements. (4%)

2.(20%) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, with $\mathbf{v}_1 = 1+x+x^2$, $\mathbf{v}_2 = 1-2x$, $\mathbf{v}_3 = x+3x^2$, $\mathbf{v}_4 = 1+x^2$, $\mathbf{v}_5 = -2-x-x^2$, be a set of five vectors in $P_2(\mathbf{R})$,

(a). Please show that $Span(S) = P_2(\mathbf{R})$. (10%)

(b). Please find a subset of S which forms a basis of $P_2(\mathbf{R})$. Please explain why your choice is a basis. (10%)

3. (25%) Let $V = M_{n\times n}(\mathbf{R})$ be the space of n by n matrices. Consider the subset W_n of skew-symmetric matrices, i.e. the set of $n \times n$ matrices m in V such that $m^t = -m$. (m^t means the transpose of m)

(a). Show that W is a subspace. (10%)

(b). For $W_3 \subset \mathbf{M}_{3\times 3}(\mathbf{R})$, find a basis of W_3 and justify that it is a basis of W_3. (10%)

(c). Determine the dimension of W_3. (5%)

4.(20%) (a). Consider a few functions

$T_1 : \mathbf{R}^2 \mapsto \mathbf{R}^3$, $T_1(x,y) = (x+|y|, y, x-y)$.
$T_2 : \mathbf{R}^2 \mapsto \mathbf{R}^3$, $T_2(x,y) = (2x+1, 4y-3, x)$.
$T_3 : P_2(\mathbf{R}) \mapsto P_4(\mathbf{R})$, $T_3(f) = f^2, f \in P_2(\mathbf{R})$.
$T_4 : \mathbf{M}_{2\times 2}(\mathbf{R}) \mapsto \mathbf{R}$, $T_4(m) = tr(m)$, for $m \in \mathbf{M}_{2\times 2}(\mathbf{R})$.
$T_5 : \mathbf{M}_{3\times 3}(\mathbf{R}) \mapsto \mathbf{M}_{3\times 3}(\mathbf{R})$ defined by $T_5(m) = m - m^t$ for $m \in \mathbf{M}_{3\times 3}(\mathbf{R})$.

List all those which are not linear transformations. For each of them, explain why it is not a linear transformation. (12%)

(b). Let $\{v_1, v_2, \cdots, v_n\}$ be a linear dependent subset of V and let $T : V \longrightarrow W$ be a linear transformation. Show that $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a linear dependent subset of W. (8%)

5.(20%) Let $S = \{v_1, v_2, \cdots, v_n\}$ be a basis of a vector space V.

(a). Prove that (without applying theorems) for any $v \in V$, but $v \notin S$, the union $S \cup \{v\}$ is linearly dependent. (10%)

(b). Prove that (without applying theorems) for any $v \in S$, the deletion of v from S, $S - \{v\}$, does not generate V. (10%)

Extra Credit:

(10%) Let $W \subset V$ be a subspace of the finite dimensional vector space V. Prove that $dim(W) \le dim(V)$.

The 8 Axioms of Vector Spaces over $\mathbf{R}$

Let V be a set with an addition $+$ and a scalar multiplication $\cdot$. V is said to be a vector space over $\mathbf{R}$ if

(VS1) For all $x, y \in V$, $x + y = y + x \in V$.

(VS2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$.

(VS3) There exists an element in V denoted by $\mathbf{0}$ such that $x + \mathbf{0} = x$ for each $x \in V$.

(VS4) For each $x \in V$, $\exists y \in V$ such that $x + y = \mathbf{0}$.

(VS5) For all $x \in V$, $1 \cdot x = x$.

(VS6) For each pair of $a, b \in \mathbf{R}$ and each element $x \in V$, $(ab) \cdot x = a \cdot (b \cdot x)$.

(VS7) For each $a \in \mathbf{R}$ and each pair of $x, y \in V$, $a \cdot (x + y) = a \cdot x + a \cdot y$.

(VS8) For each pair of $a, b \in \mathbf{R}$ and each element $x \in V$, $(a + b) \cdot x = a \cdot x + b \cdot x$.