P. Vojta
Fall 2003

Math 113 Final Examination

3 hours

1. (20 points) Carefully define the following. (In each definition you may use without defining them any terms or symbols that were used in the text prior to that definition.)

   (a). normal subgroup

   (b). inner automorphism

   (c). group action

   (d). isotropy subgroup

   (e). ideal

2. (20 points) Carefully define the following, under the same conditions as the first problem.

   (a). unit

   (b). irreducible (element of an integral domain)

   (c). constructible number

   (d). α is algebraic over a field F

   (e). algebraically closed field

3. (25 points) Compute the following:

   (a). The center of the group given by the table:

*	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	b	c	e	g	h	f	d
b	b	c	e	a	f	d	h	g
c	c	e	a	b	h	g	d	f
d	d	h	f	g	e	b	c	a
f	f	g	d	h	b	e	a	c
g	g	d	h	f	a	c	e	b
h	h	f	g	d	c	a	b	e

   (b). The characteristic of the ring $\mathbb{Z}_6 \times \mathbb{Z}_{20}$.

   (c). The last digit of the number $23^{123,321,123,321,268,862}$.

4. (15 points) Determine all subgroups of $\mathbb{Z}_5 \times \mathbb{Z}_5$, and draw a subgroup diagram of them.

5. (25 points) Prove Cayley's theorem, that every group is isomorphic to a group of permutations. At your option, your proof may use material from later in the book (but not Cayley's theorem itself, of course).

6. (25 points) Factor the polynomial

$$10x^4 - 10x^3 - 490x^2 + 560x - 70$$

into irreducibles in $\mathbb{Z}[x]$. Give some justification for why each factor is irreducible.

7. (25 points) (a). Show that the zero ideal in $\mathbb{Z}[x]$ is prime but not maximal. [*Hint:* There's an easy way and a hard way to do this.]

   (b). Find another ideal in $\mathbb{Z}[x]$ that is prime but not maximal.

   (c). Show that all nonzero prime ideals in $\mathbb{Z}[i]$ are maximal.

8. (20 points) Let E/F be a field extension. Let $\alpha \in E$ be algebraic of odd degree over F. Show that α^2 is algebraic of odd degree over F, and that $F(\alpha) = F(\alpha^2)$.