MATH 222A – FINAL EXAM F'03 EVANS

INSTRUCTIONS: This test is due in my office, 907 Evans Hall, by 4 pm on Tuesday, December 9. You may refer to the textbook and your own class notes, but do not consult any other book or article or any person. Please come by my office if you have any questions about these problems.

Problem #1. The motion of an elastic solid in $\mathbb{R}^3$ is modeled by the *system* of PDE

$$(*) \qquad \rho u^i_{tt} = \mu \Delta u^i + (\lambda + \mu)(\operatorname{div} \mathbf{u})_{x_i} \qquad (i = 1, 2, 3),$$

where ρ, μ, λ are positive constants.

(i) Show that if $\mathbf{u} = (u^1, u^2, u^3)$ solves $(*)$, then

$$(\operatorname{div} \mathbf{u})_{tt} = \frac{\lambda + 2\mu}{\rho} \Delta(\operatorname{div} \mathbf{u}) \quad \text{and} \quad (\operatorname{curl} \mathbf{u})_{tt} = \frac{\mu}{\rho} \Delta(\operatorname{curl} \mathbf{u}).$$

(ii) One sometimes says that $(*)$ supports disturbances moving with either of the two characteristic velocities

$$\left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}} \quad \text{and} \quad \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}.$$

Give a careful mathematical interpretation.

Problem #2. Compute explicitly the unique entropy solution of

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ \qquad\qquad u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1. \end{cases}$$

Draw a picture documenting your answer, being sure to illustrate what happens for all times $t > 0$.

Problem #3. Prove the following Sobolev–type inequality: There exists a constant C such that for all functions $u \in W^{2,1}(\mathbb{R}^2)$ we have

$$(**) \qquad ||u||_{L^\infty(\mathbb{R}^2)} \leq C||u||_{W^{2,1}(\mathbb{R}^2)}.$$

Hints: First note carefully that $W^{2,1}$ denotes the Sobolev space of functions in L^1 that have weak first and second partial derivatives belonging to L^1. To prove $(**)$, we may first assume u is smooth and has compact support. Fix any point $x = (x_1, x_2) \in \mathbb{R}^2$. Then

$$u(x) = \int_{-\infty}^{x_1} u_{x_1}(s, x_2)\, ds.$$

Continue the proof from here, showing $|u(x)| \leq C||D^2 u||_{L^1(\mathbb{R}^2)}$. Explain then how to prove $(**)$ if u is not smooth, with compact support.