Math H53 Final Exam

December 16th, 2003

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1: (5 pts) By use of Tangent Planes, estimate $f(1.01, \pi + 0.02)$ where $f(x, y) = x^2 \sin(y) + \cos(x\ y)$.

2 a: (5 pts) Find a parametric representation for the intersect of $x^2 + z^2 = 1$ and $x^2 + y^2 = 1$.

b: (5 pts) Let
$$f(x, y) = x^2 + y^2 + \mathrm{e}^{-x^2 - y^2}$$
Find all local minimums and maximums.

3: (5 pts) Let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$ be unit vectors in $\mathbb{R}^2$. Assume that f has continuous partials of every order. Show, by means of Clairut's Theorem, that the directional derivatives
$$f_{v,u}(x, y) = f_{u,v}(x, y)$$

4: (5 pts) Let $f(x, y, z) = x^2$. Find the maximum of f on the intersection of the two surfaces $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

5: (5 pts) Prove that the rational numbers $\mathbb{Q}$ have measure 0.

6: (5 pts) Find the volume of the region bounded by the function $z = f(x, y) = 3 + x^2 - x^2 \sin(yx)$ over $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$

7: (5 pts) Assume that e, f, g and h are continuous functions. Show that $F(u, x, y, z) = \langle e(u), f(x), g(y), h(z) \rangle$ is conservative. [Hint, consider the Fundamental Theorem of Calculus.] Find an example where $\langle f(y), g(x) \rangle$ is not conservative.

8: (5 pts) Let $z = f(x, y)$. By using the general equation for the surface area of a parametric curve, show that the surface area of f over D is
$$\iint_D \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA$$

9: (5 pts) Prove that the line integral $\int_C f \cdot dr = 0$ for all closed curves C if and only if $\int_P f \cdot dr$ is path independent for all curves P. (Do this directly, do not use the fact that f is conservative.)

10: (5 pts) Let $f(x,y) = \left[\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right]$. Show $\int_C f \cdot dr = 0$ for all closed curves C, so long as C does not contain the point $(0,0)$ in the interior.

11 a: (5 pts) State Greens Theorem

b: (5 pts) Let $D = \{(x,y) : x^2 + y^2 \leq 1$. Let $F(x,y) = \langle P(x,y), 0\rangle$ Prove Green's Theorem for this simplified case.

12 a: (5 pts) State Stokes Theorem.

b: (5 pts) Use Stokes Theorem to evaluate $\iint_S \text{curl} F \cdot dS$ where $F(x,y,z) = \langle yz, xz, xy\rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 3$, oriented upwards.

13 a: (5 pts) State the Divergence Theorem

b: (5 pts) Give an example of the following

i: A disconnected unbounded 1-manifold.

ii: A closed bounded 2-manifold.

iii: A 3-manifold.

14: (5 pts)

a: Convert $(x,y) = (1,2)$ from Cartesian (rectangular) co-ordinates to polar co-ordinates.

b: Convert $(r,\theta) = (1, \frac{\pi}{4})$ from polar co-ordinates to Cartesian co-ordinates.

c: Convert $(x,y,z) = (1,1,1)$ from Cartesian co-ordinates to cylindrical co-ordinates.

d: Convert $(x,y,z) = (1,1,\sqrt{2})$ from Cartesian co-ordinates to spherical co-ordinates.

e: Convert (r,θ,z) from cylindrical co-ordinates to spherical co-ordinates.