Fall 2003, Math 110-8 – Linear Algebra. Name (PRINT): ______________________

Instructor: Ilan Hirshberg Time: 3 hours

Final Examination

Instructions:

(1) This exam has two parts. In each part, you should answer 4 questions out of the given 5. Part A is worth 40 points (10 points per question), and part B is worth 60 points (15 points per question). The maximum number of points on the exam is 100.
Do not answer more questions than asked – if you do, some of your answers will be ignored at random.

(2) You may use theorems proved in class, unless they are essentially the same as what you are asked to prove, provided that you give a *full and correct statement of the theorem* and *explain how it is being used.*

(3) Credit may be removed for irrelevant, incoherent or illegible information. Cross out your scratchwork.

(4) The last page is scratch paper, included for your convenience. Feel free to detach it – you do not need to turn it in.

Good luck!

Email notification:
If you want your scores emailed to you, please sign here: ______________________
and make sure that I have your email address.

Part A

1	
2	
3	
4	
5	

Part B

1	
2	
3	
4	
5	

Total: /100

PART A: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(1) (a) (5 points) Let V be a complex inner product space. Show that the inner product can be recovered from the norm (give an explicit formula for $\langle v, w \rangle$ as an expression involving norms of various vectors from V, and show that your formula is indeed correct).

(b) (5 points) Prove that there is no inner product on $\mathbb{C}^2$ such that the norm associated to the inner product is the ℓ^∞ norm.

PART A: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(2) Let V, U be a vector spaces over F, $W \subseteq V$ a subspace. Let $Q : V \to V/W$ be the transformation $Qv = [v]$ (where $[v]$ denotes the equivalence class of v). Prove that for any linear transformation $T : V \to U$ such that $ker(T) \supseteq W$ there exists a linear transformation $S : V/W \to U$ such that $T = S \circ Q$ (to receive full credit, you must (a) explain how to define S, (b) prove that your S is well defined, and (c) prove that S is a linear transformation).

PART A: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(3) Let T be a linear transformation from a finite dimensional vector space V to itself. Suppose that the characteristic polynomial of T is $p(x) = (1-x)^5(2-x)^3$, the minimal polynomial is $m(x) = (x-1)^3(x-2)^2$, and $null(T - id_V) = 3$. Write down a matrix J in Jordan form which represents T with respect to some basis (and explain how you obtained the matrix).

PART A: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(4) Let V be a vector space (over a field F), and let v_1, v_2, v_3 be a basis for V. Let W be a two dimensional subspace of V. Is it necessarily the case that there is a pair of vectors among the three vectors v_1, v_2, v_3 which forms a basis for W? Either prove this, or provide a counterexample (and justify it).

PART A: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(5) Consider $\mathbb{C}^3$ as an inner product space, with the standard inner product. Suppose that the linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ satisfies: $T(1,1,0) = (0,-1,1)$, $T(1,0,0) = (-1,0,1)$, $T(0,0,1) = (0,1,-1)$. Let $W = ker(T^*)$. Compute $det(T|_{W^\perp})$. (You do not need to prove that $W^\perp$ is invariant for T).

PART B: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(1) State and prove the rank and nullity theorem.

PART B: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(2) Let W, U be two finite dimensional subspaces of a vector space V. Prove that $dim(W+U) = dim(W) + dim(U) - dim(W \cap U)$.

Part B: answer 4 questions out of the given 5

(3) Let V be a finite dimensional complex inner product space, and let $T : V \to V$ be a self-adjoint linear transformation.

(a) (5 points) Prove that if W is an invariant subspace for T, then so is $W^{\perp}$.

(b) (5 points) Prove that T has an orthonormal eigenbasis.

(c) (5 points) Prove that all the eigenvalues of T are real.

(Note: Parts (a),(b) are special cases of theorems we proved in class – you are expected to do those problems without using those theorems, unless you state and prove them first).

PART B: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(4) Let V be a vector space, and let $T : V \to V$ be a linear transformation.
   (a) (10 points) Suppose a given vector v satisfies $T^k v = 0$ but $T^{k-1} v \neq 0$ (where k is a positive integer). Prove that $v, Tv, ..., T^{k-1}v$ are linearly independent.
   (b) (5 points) Suppose V is finite dimensional, and T is nilpotent, with nilpotence index $n = dim(V)$. Prove that there is a basis $v_1, ..., v_n$ for V such that $Tv_1 = 0$, and $Tv_j = v_{j-1}$ for $j = 2, 3, ..., n$.

PART B: ANSWER 4 QUESTIONS OUT OF THE GIVEN 5

(5) Let V be a finite dimensional real inner product space, and let $T : V \to V$ be a linear transformation.

(a) (5 points) Prove that T is orthogonal if and only if $\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in V$.

(b) (5 points) Prove that if $dim(V)$ is odd then T has an eigenvalue.

(c) (5 points) Suppose that $dim(V)$ is odd and T is orthogonal. Prove that there is a non-zero vector $v \in V$ such that $T^2 v = v$.