MATH 113: INTRODUCTION TO ABSTRACT ALGEBRA (Section 4)

Midterm 2

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Be sure to justify your answers. You may use any preceding parts to answer each question. Good luck!

1. For each group, determine whether it is simple or not. Justify your answers.
   (a) (*8 Points*) A group of prime order.
   (b) (*8 Points*) A group of order 42. (**Hint**: Consider the Sylow 7-subgroups.)

2. You may answer the following two questions without any justification.
   (a) (*4 Points*) Let p be a prime. List all groups of order p^2, up to isomorphism.
   (b) (*4 Points*) List all groups of order less than or equal to 7, up to isomorphism.

3. Let n be a positive integer and consider the ring $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ under the congruence addition and multiplication modulo n.
   (a) (*8 Points*) In the ring $\mathbb{Z}_n$, show that any number a coprime to n is a unit.
   (b) (*8 Points*) Let $G_n \subset \mathbb{Z}_n$ be the subset of all elements $a \in \mathbb{Z}_n$ such that $\mathrm{GCD}(a, n) = 1$. Prove that G_n is a group under the congruence multiplication modulo n.
   (c) (*8 Points*) Let G_n and $\mathbb{Z}_n$ be as in (b). Find G_8 and describe to which group it is isomorphic.

4. (a) (*10 Points*) Let G be a *non-abelian* group and $Z(G)$ its center. Prove that the factor group $G/Z(G)$ is not a cyclic group.
   (b) (*10 Points*) Prove that a non-abelian group of order pq, where p and q are distinct primes, has a trivial center.

5. (a) (*8 Points*) Let $D_1 \neq \{0\}$ and $D_2 \neq \{0\}$ be integral domains. Show that the direct product ring $D_1 \times D_2$ cannot be an integral domain. (**Hint**: Find a zero-divisor.)
   (b) (*8 Points*) In the ring $\mathbb{Z}_8$, find the zero-divisors and find the roots of the equation $x^2 - 4x + 3 = 0$.
   (c) (*8 Points*) Find the characteristic of the ring $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{10}$.
   (d) (*8 Points*) An element a in a ring is called idempotent if $a^2 = a$. Let R be a ring with unity $1 \neq 0$ and with no zero-divisors. Prove that R has precisely two idempotent elements.