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Fall 2002, Math 113, Sec. 5

Second Midterm

28 Oct., 2002
3:10-4:00

1. (27 points, 9 points each.) Find the following. If the answer to a question is a set, you should give it by listing or describing its elements in set brackets, { ... }.

(a) The kernel of the homomorphism from $\mathbb{Z}$ to D_{10} (the group of symmetries of a pentagon) taking each $n \in \mathbb{Z}$ to rotation by $n(4\pi/5)$ radians.

(b) The coset of A_3 in S_3 that contains $(1\ 2)$.

(c) The number of fixed points of σ^3, if σ is an element of S_n whose complete cycle decomposition consists of a cycles of length 3, b cycles of length 2, and $n-3a-2b$ cycles of length 1.

2. (36 points; 9 points each.) For each of the items listed below, either *give an example,* or give a brief reason why *no example exists.* (If you give an example, you do *not* have to prove that it has the property stated. Examples should be specific for full credit; i.e., even if there are many objects of a given sort, you should name one.)

(a) A simple non-cyclic group.

(b) A subgroup of $\mathbb{Z} \times \mathbb{Z}$ that is not normal.

(c) An injective (i.e., one-to-one) homomorphism $f : \mathbb{Z} \to \mathbb{R}^{\times}$. (Recall that $\mathbb{R}^{\times}$ denotes the group of nonzero real numbers under multiplication.)

(d) A group G and a subgroup H, such that H is not the kernel of any homomorphism with domain G.

3. (14 points.) Let G and H be groups, $f : G \to H$ an injective (i.e., one-to-one) homomorphism, and $g \in G$ an element of finite order n. Show that $f(g)$ also has order n.

4. (14 points.) Let G be a group. Recall that $Z(G)$, the center of G, means $\{z \in G : \forall\, g \in G,\ zg = gz\}$. Show that $Z(G)$ is a subgroup of G. (Rotman describes this as "easy to see". I am asking you to supply the details.)

5. (9 points.) Let G be a group which acts on a set X, and let $x, y \in X$. Show that if $\mathcal{O}(x)$ and $\mathcal{O}(y)$ have an element in common, then they are equal. (Recall that $\mathcal{O}(x)$ denotes $\{gx : g \in G\}$. The result you are to prove is part of a result proved by Rotman, that X is the disjoint union of the orbits. Hence you may not call on that result in proving this.)