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MATHEMATICS 74: FINAL EXAM
December 18, 2002

Follow all of the instructions carefully. Make sure to give reasons if they are asked for. Good luck!!!

Question I. (3 pts each)
Let A, B and C be sets, and let $f : A \to B$ and $g : B \to C$.

1) If f and g are onto, prove that $g \circ f$ is onto.

2) If $g \circ f$ is onto, prove that g must be onto.

3) If g is onto, is it true that $g \circ f$ must be onto? If so, then prove it, and if not, then give a counter-example (specifying A, B, C, f, and g).

4) Suppose that $g \circ f$ is 1-1 and that f is onto. Prove that g must be 1-1.

Question II.
Determine whether each of the following statements is true or false. Give a brief reason. (2 pts each)

1) $\exists x \in \mathbb{N}\ \forall y \in \mathbb{N}\ (y < x \Longrightarrow y \text{ is prime})$ (Recall that 1 is not a prime.)

2) For any sets X and Y, if $X \cap Y = \emptyset$, then $\mathcal{P}(X) \cap \mathcal{P}(Y) = \emptyset$.

3) There is exactly one function from $\{1\}$ to $\emptyset$.

4) $\{(4,1),(5,1)\}$ is a transitive relation.

5) $\{\emptyset,\ \{\emptyset\},\ \{\emptyset,\{\emptyset\}\}\} \approx \{1,-\pi,7\}$

6) Every 1-1 function from $\mathbb{Z}$ to $\mathbb{Z}$ is onto.

Question III.
Let $A = \mathcal{P}(\mathbb{R})$ and consider the relation S on A defined by:

$$(X,Y) \in S \iff X - Y = \emptyset.$$

1) (1 pt) Give an example of a set Z such that $([1,2], Z) \in S$.

2) (3 pts) Find the domain of S. Prove your answer.

3) (4 pts)Is S reflexive? Is S symmetric? Give a proof or a counter-example.

4) (3 pts) Prove that S is transitive. That is, suppose that X, Y and Z are subsets of $\mathbb{R}$; prove that if $X - Y = \emptyset$ and $Y - Z = \emptyset$, then $X - Z = \emptyset$.

Question IV.
Proofs to Grade: Carefully evaluate each of the following arguments as usual: If the argument correctly proves the claim, give the proof an A. If there is a flaw in the reasoning, or a statement that doesn't really make sense, determine exactly where the argument breaks down, and give it a C (if a small change will yield a correct proof of the claim) or an F (if not). (3 pts each)

1) Claim: Let A, B and C be sets. If $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$, then $A \cap C \neq \emptyset$.

"Proof": Since $A \cap B \neq \emptyset$, there is some $x \in A \cap B$, so $x \in A$. Since $B \cap C \neq \emptyset$, there is $x \in B \cap C$, which implies that $x \in C$. Since $x \in A$ and $x \in C$, we conclude that $x \in A \cap C$; therefore, $A \cap C \neq \emptyset$.

2) Assume that π is irrational. Then for any $x \in \mathbb{R}$, either $\pi - x$ is irrational, or $\pi + x$ is irrational.

"Proof": We assume that π is irrational, so there are no integers a and b such that $\pi = \frac{a}{b}$. Consider $x = \pi$. Then $\pi - x = 0$, which is rational. On the other hand, $\pi + x = 2\pi$. Now suppose, by way of contradiction, that 2π were also rational. Then there would be integers c and d such that $2\pi = \frac{c}{d}$. But then $\pi = \frac{c}{2d}$, and so π is rational, which is a contradiction. So either $\pi - x$ or $\pi + x$ is irrational.

Question V.

1) (3 pts) Let $X \subseteq \mathbb{R}$. Suppose that for all $W \subseteq \mathbb{R}, X \cap W = W$. Prove that $X = \mathbb{R}$.

2) (3 pts) Let $X \subseteq \mathbb{R}$. Suppose that for all $W \subseteq \mathbb{R}, X \cap W = X$. What set must X be? Prove your answer.

Question VI. (6 pts)

Let X and Y be countably infinite sets. Prove that $X \times Y$ is countable. (You may be informal in your presentation.)

Question VII. (12 pts)
Do one of the following problems. If you attempt both, make sure it's clear to me which one you want me to grade by crossing one of the attempts out.

Note: Your first instinct may be to do the proof to grade. I don't recommend chosing this option unless you are fairly sure of your answer. This is because it will be difficult to get any partial credit on the proof to grade!

Option i) For each $n \in \mathbb{N}$, let X_n be the set of all functions $f : \{1, ..., n\} \to \mathbb{N}$. Use induction to prove that for each $n \in \mathbb{N}$, X_n is countable. You should give a careful and complete proof, but you are allowed to use any results from class.

Option ii) Grade the following argument. I don't care about "C" vs "F" here; just mark the proof "correct" or "incorrect". If your answer is "incorrect", specify the exact place where the argument goes wrong.

Claim: Let X be a finite set, and let $f : X \to X$. If f is 1-1, then f has finite order; that is, there is a $k \in \mathbb{N}$ such that $f^k(x) = x$ for all $x \in X$.

"Proof": Let P_n be the statement: Whenever X has n elements, and f is a 1-1 function from X to itself, then f has finite order. We will use strong induction to prove P_n true for all $n \in \mathbb{N}$.

base case: If X has one element, there is only one function $f : X \to X$. Clearly $f(x) = x$ for all $x \in X$, so f has finite order.

inductive step: Let $n \geq 2$. We assume P_m holds for all $m < n$, and show that P_n holds. So let X be a set with n elements, and let $f : X \to X$ be 1-1. Split X up into two disjoint subsets Y and Z, as close to the same size as possible. (For instance, if n is even, say $n = 2l$, then let Y and Z both have l elements; if n is odd, say $n = 2l + 1$, then let Y have l elements and Z have $l + 1$ elements.) Let g be the "restriction" of f to Y; that is, $g = \{(x, f(x)) : x \in Y\}$. Similarly, let $h = \{(x, f(x)) : x \in Z\}$.

Note that Y and Z each have less than n elements. Furthermore, since f is 1-1, clearly g and h are 1-1 too. So by strong induction hypothesis, g and h have finite orders. This means that there are k_1 and $k_2 \in \mathbb{N}$ such that $g^{k_1}(x) = x$ for all $x \in Y$, and $h^{k_2}(x) = x$ for all $x \in Z$. Now, let $k = k_1 k_2$. I claim that $f^k(x) = x$ for all $x \in X$. Pick such an x; then either $x \in Y$, or $x \in Z$. Suppose $x \in Y$ (the other case is similar). Then $f^k(x) = g^k(x)$, since g does the same thing that f does on elements of Y. But applying the function g k times is the same as applying the function g^{k_1} k_2 times, since $k = k_1 k_2$. Every time we apply g^{k_1}, we get back to x. Therefore, $g^k(x) = x$. So $f^k(x) = x$. This proves statement P_n.