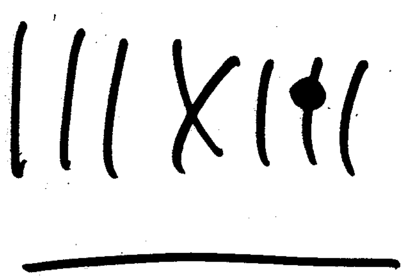
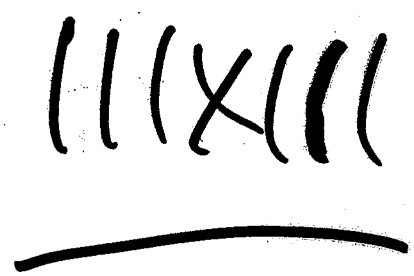
(*)

AS WE EXPLAINED LAST TIME, GEOMETRIC LANGLANDS COMES FROM MIRROR SYMMETRY AMONG BRANES



$\mathcal{M}_H({}^LG, C)$

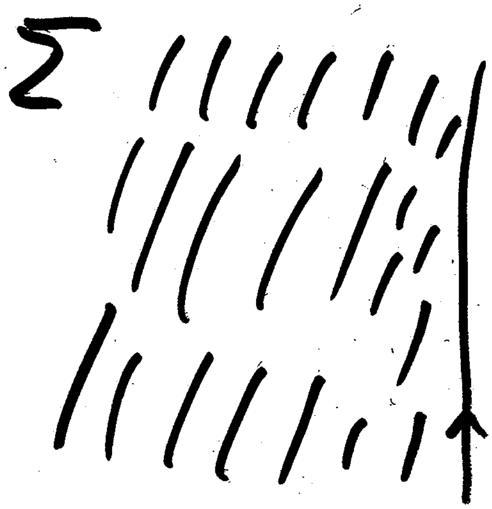
B - BRANE



$\mathcal{M}_H(G, C)$

A - BRANE

CONCRETELY THE A-BRANE OR B-BRANE IS A BOUNDARY CONDITION IN THE QUANTUM THEORY



EXAMPLE: B-BRANE

DIRICHLET BOUNDARY CONDITIONS

$$\Phi\big|_{\partial\Sigma} : \partial\Sigma \to \text{GIVEN POINT IN } \mathcal{M}_H$$

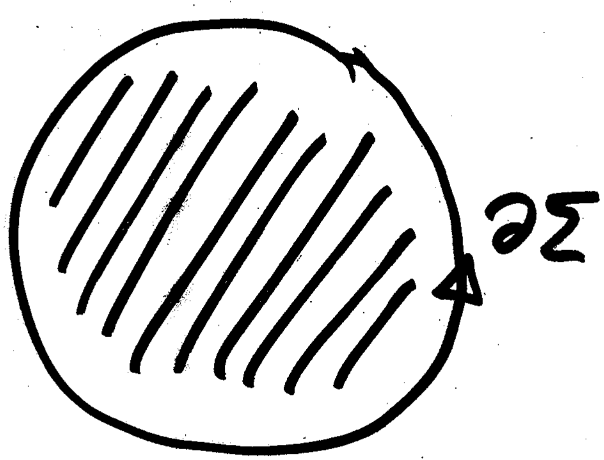
MORE GENERALLY

$$\Phi|_{\partial\Sigma} : \partial\Sigma \to Y \subset \mathcal{M}_H$$

Y ENDOWED WITH VECTOR BUNDLE $\mathcal{V}$ WITH CONNECTION

PATH INTEGRAL HAS A FACTOR

$$\int \mathcal{D}\Phi \ldots \exp(-I) \; \underbrace{\mathrm{Tr}\, P\exp\left(-\int_{\partial\Sigma} \mathcal{B}\right)}_{\text{HOLONOMY AROUND } \partial\Sigma}$$



THE SUBMANIFOLD Y AND ITS VECTOR BUNDLE V OBEY APPROPRIATE CONDITIONS

<u>B-MODEL</u> THEY DEFINE A COHERENT SHEAF → X
MH

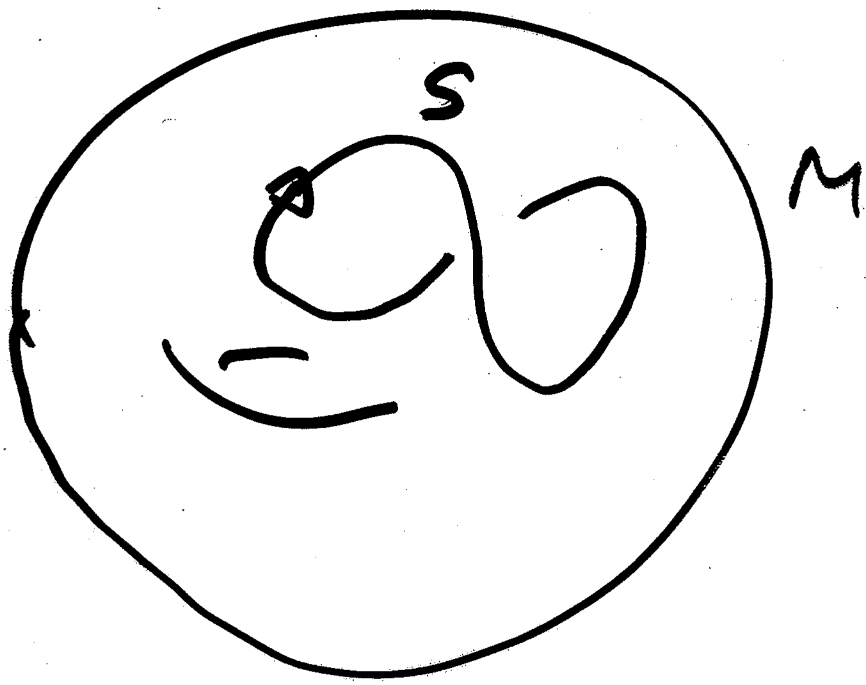
<u>A-MODEL</u> STANDARD EXAMPLE

Y = LAGRANGIAN AND V IS FLAT

PLUS OTHER EXAMPLES IN GENERAL
("CO ISOTROPIC BRANES" - KAPUSTIN & ORLOV)

"LINE OPERATORS" IN FOUR DIMENSIONS

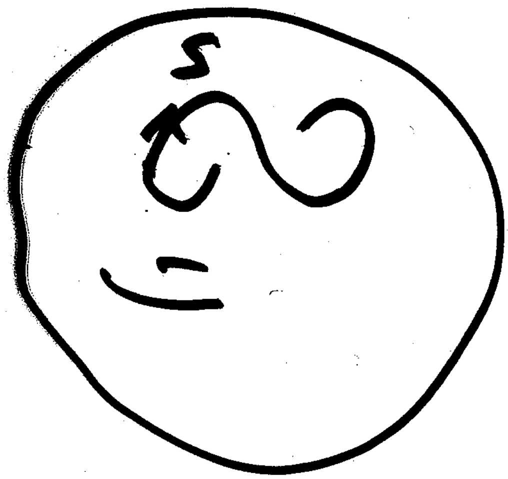
WILSON OPERATOR = HOLONOMY IN REPRESENTATION R



$$W_R(S) = \mathrm{Tr}_R \, P\exp\left(-\int_S A\right)$$

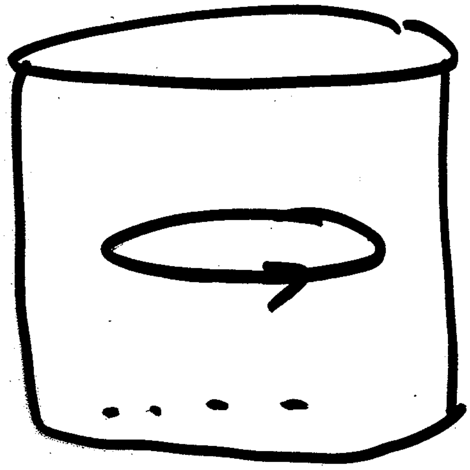
= trace of holonomy in R - rep.

WE INCLUDE THE "WILSON OPERATOR" AS A FACTOR IN PATH INTEGRAL



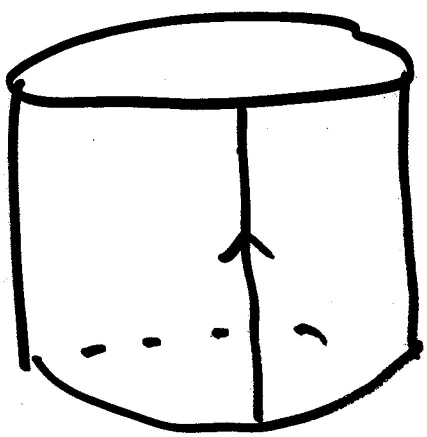
$$\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\, \mathcal{D}\ldots \exp(-I) \cdot W_R(S)$$

IT IS CALLED AN "OPERATOR" BECAUSE IN A CERTAIN SITUATION IT IS UPON QUANTIZATION A HILBERT SPACE OPERATOR



↑ TIME

WE'LL ACTUALLY BE INTERESTED IN THE OPPOSITE SITUATION



DON'T TAKE TRACE OF THE HOLONOMY

JUST USE HOLONOMY OPERATOR FROM INITIAL TO FINAL STATE.

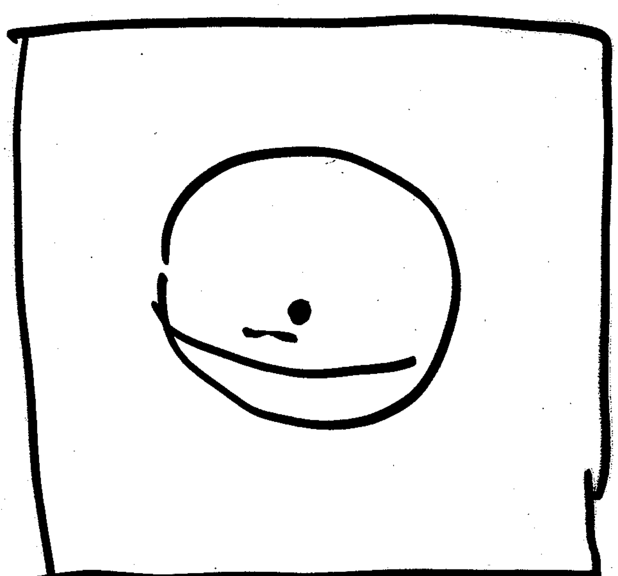
THE "OPERATOR" IS REALLY PART OF THE DEFINITION OF THE QUANTUM PROBLEM.

BUT WHAT DOES A "WILSON OPERATOR" TRANSFORM INTO UNDER S-DUALITY?

ANSWER (LATE 1970's):

AN "'t HOOFT OPERATOR"

TO DEFINE IT, START WITH THE "MAGNETIC MONOPOLE" SOLUTION OF MAXWELL'S EQUATIONS ON $\mathbb{R}^3 \setminus 0$



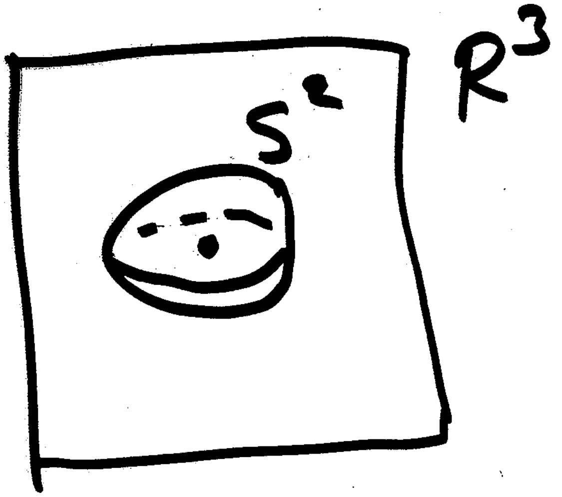
$$F = \frac{1}{2} * d\left(\frac{1}{|x|}\right)$$

S^2

$$\int_{S^2} F = 2\pi$$

THE COEFFICIENT IS CHOSEN

SO THAT, RESTRICTED TO S^2



F IS THE CURVATURE OF A CONNECTION ON

$$\mathcal{O}(1) \to S^2 \simeq \mathbb{C}P^1$$

NOW WE PICK A REPRESENTATION $^L\rho$ OF ^LG AND A CORRESPONDING

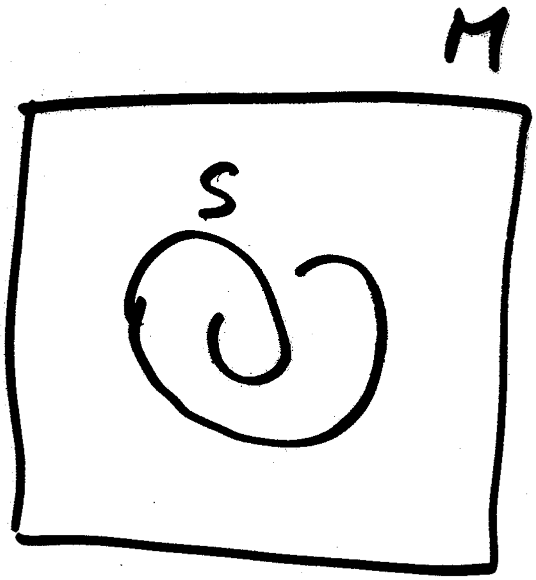
$$^L\rho : U(1) \to T \hookrightarrow G$$

THIS GIVES A SOLUTION OF YANG-MILLS EQN'S FOR G.

NOW DEFINE THE 't HOOFT OPERATOR $T(S; {}^{L}R)$

IN G GAUGE THEORY

$E \to M/S$



BY SAYING THAT NEAR S THE FIELDS MUST HAVE THE SINGULARITY DEFINED BY ${}^{L}\rho$:

$$\left.\begin{array}{l}\displaystyle\int_{(\mathcal{A}x\ldots)_{{}^{L}\rho}} \mathcal{D}A \ldots \exp -I\end{array}\right\}$$

THE "USUAL" PATH INTEGRAL, BUT WE INTEGRATE OVER A DIFFERENT SET OF FIELDS

THE S-DUALITY CONJECTURE

SAYS THAT

A WILSON OPERATOR

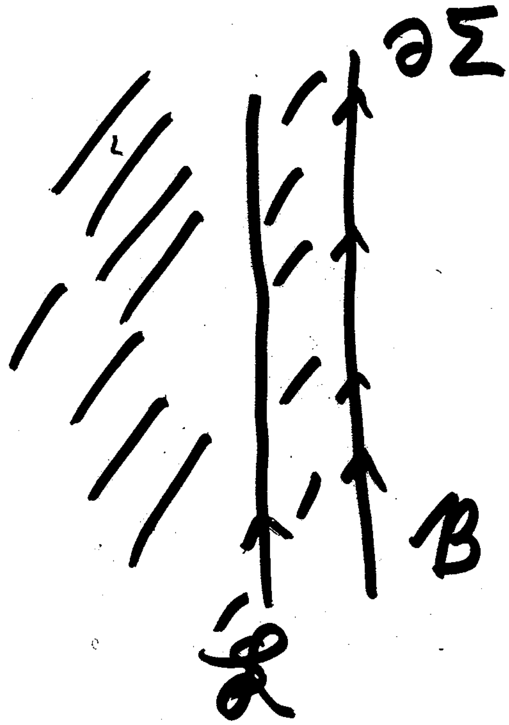
$W(S, {}^L R)$ IN ${}^L G$ GAUGE

THEORY MAPS TO AN 't HOOFT

OPERATOR $T(S, {}^L R)$

IN G GAUGE THEORY.

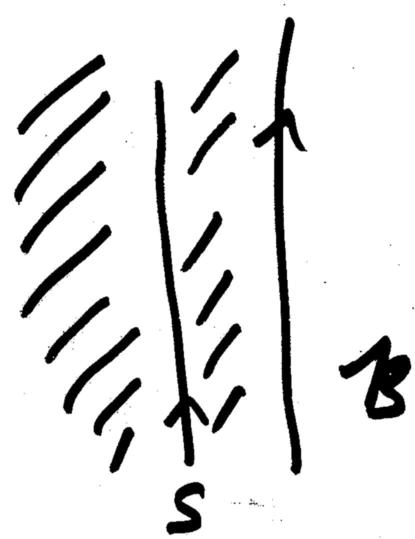
$\mathcal{L} = W$ OR T



NOW REGARDLESS OF WHETHER OUR LINE OPERATOR $\mathcal{L}$ IS A WILSON OR 't HOOFT OPERATOR, THINK OF IT RUNNING PARALLEL TO THE BOUNDARY AS SHOWN.

THE BOUNDARY IS LABELED BY A BRANE $\mathcal{B}$.

THE COMPOSITE OF THE LINE OPERATOR $\mathcal{L}$ AND BRANE $\mathcal{B}$ IS EQUIVALENT TO A NEW BRANE THAT WE CALL $\mathcal{B}'$



WE WRITE

$$\mathcal{B}' = \mathcal{L} \cdot \mathcal{B}$$

BRANES FORM A "CATEGORY"

GIVEN $\mathcal{B}_1, \mathcal{B}_2$ WE DEFINE

A "SPACE OF MORPHISMS"

$H_{(\mathcal{B}_1, \mathcal{B}_2)}$

A LINE OPERATOR $\mathcal{L}$

GIVES A MORPHISM

FROM THE CATEGORY

OF BRANES TO ITSELF

$$\mathcal{B} \rightarrow \mathcal{L}\,\mathcal{B} = \mathcal{B}'$$

WE SAY $\mathcal{B}$ IS AN "EIGENBRANE" FOR $\mathcal{L}$ IF $\mathcal{B}'$ IS A "MULTIPLE" OF $\mathcal{B}$, WHICH MEANS

$$\mathcal{B}' = \mathcal{B} \otimes U$$

U A FIXED VECTOR SPACE

THIS MAKES SENSE ABSTRACTLY

$$\mathcal{H}_{(\tilde{\mathcal{B}},\mathcal{B}')} = \mathcal{H}_{(\tilde{\mathcal{B}},\mathcal{B})} \otimes U$$

FOR ALL $\tilde{\mathcal{B}}$.

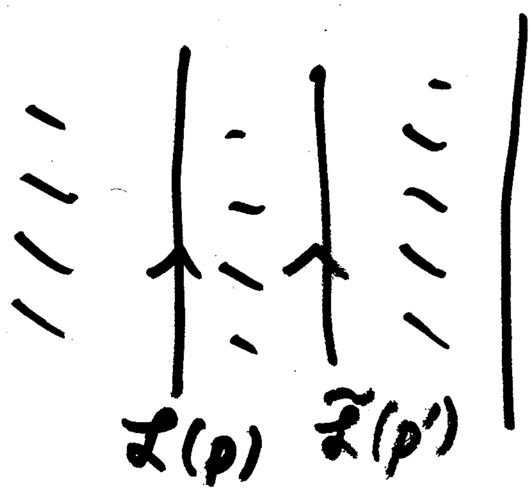
BUT IN GEOMETRY IT MEANS THAT THE SHEAVES $\mathcal{V}'$, $\mathcal{V}$ DEFINING $\mathcal{B}'$, $\mathcal{B}$ OBEY $\mathcal{V}' = \mathcal{V} \otimes U$

IN TOPOLOGICAL FIELD THEORY

LINE OPERATORS DERIVED

FROM FOUR DIMENSIONS

COMMUTE WITH EACH OTHER



p, p' TWO POINTS IN C

AND CAN HAVE SIMULTANEOUS

EIGENBRANES

SO AT $\Psi = \infty$, WE CAN LOOK FOR A SIMULTANEOUS EIGENBRANE FOR ALL WILSON OPERATORS

$$W(S, p, {}^L R)\, \mathcal{B} = \mathcal{B} \otimes \mathcal{U}_p({}^L R)$$

$p \in C$

FOR ALL ${}^L R$

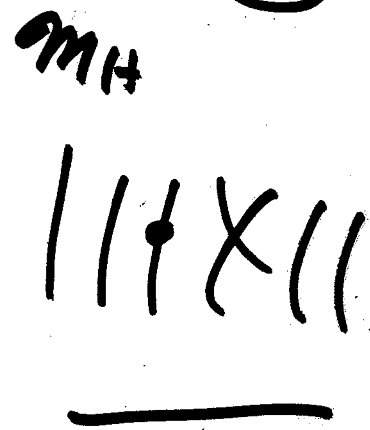
WE CALL SUCH A BRANE AN ELECTRIC EIGENBRANE.

S-DUALITY WILL MAP AN "ELECTRIC EIGENBRANE" TO A "MAGNETIC EIGENBRANE":

$$T(S, {}^{L}R)\,\tilde{\mathcal{B}} = \tilde{\mathcal{B}} \hat{\otimes} \mathcal{U}({}^{L}R)$$

WITH THE SAME "EIGENVALUE"

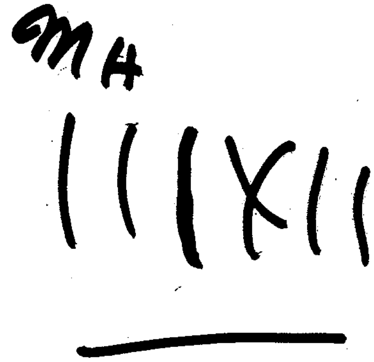
THE ZEROBRANE $\mathcal{B}$



DEFINED BY

$\rho : \pi_1(C) \to {}^L G$

IS AN ELECTRIC EIGENBRANE



SO ITS DUAL

$\tilde{\mathcal{B}}$ IS A MAGNETIC

EIGENBRANE, WHICH GEOMETERS

DESCRIBE BY TALKING ABOUT A

"HECKE EIGENSHEAF."

TO SHOW THE ZEROBRANE IS AN "ELECTRIC EIGENBRANE":

WILSON OPERATOR

$W(p; {}^LR)$

p = POINT IN C

${}^LR$ = REP OF ${}^LG$

ACTS AS FOLLOWS:

LET $\mathcal{E}_{{}^LR} \to \mathcal{M}_H({}^LG) \times C$

BE THE UNIVERSAL BUNDLE

IN REPRESENTATION ${}^LR$

AND RESTRICT IT TO

$\mathcal{M}_H \times p$

$\mathcal{M}_H \times p \qquad p \in C$

I CALL THE RESTRICTION

$$\mathcal{E}_{{}^LR}|_p \to \mathcal{M}_H$$

THEN IN GENERAL FOR A BRANE $\mathcal{B}$ DERIVED BY A SHEAF $\quad \mathcal{V} \to \mathcal{M}_H({}^LG, C)$

THE LINE OPERATOR $W(p; {}^LR)$ ACTS BY

$$\mathcal{V} \to \mathcal{V} \otimes \mathcal{E}_{{}^LR}|_p$$

SO $\mathcal{B}$ IS AN ELECTRIC EIGENBRANE PRECISELY IF

$$\mathcal{E}_{L_R}|_p$$

I.E. THE UNIVERSAL BUNDLE

$$\mathcal{E}_{L_R} \to \mathcal{M}_H \times C$$

RESTRICTED TO $\mathcal{M}_H \times p$

IS TRIVIAL IF FURTHER RESTRICTED TO THE SUPPORT OF $\mathcal{V}$

THIS IS CERTAINLY SO IF $\mathcal{V}$ IS SUPPORTED AT A POINT IN $\mathcal{M}_H$ CORRESPONDING TO $\rho : \pi_1(C) \to {}^L G_{\mathbb{C}}$

IIIXII

IN THAT CASE, $W({}^L R, p)$

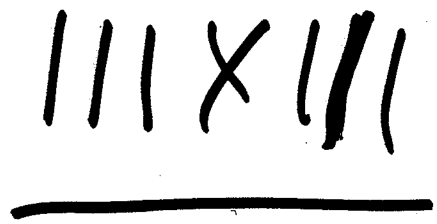
ACTS BY

$$\mathcal{V} \to \mathcal{V} \otimes \mathcal{E}_{{}^L R}|_{\rho \times p}$$

AND THE "EIGENVALUE" IS THE FIBER AT p OF THE FLAT BUNDLE $\mathcal{U}_{{}^L R} \to C$

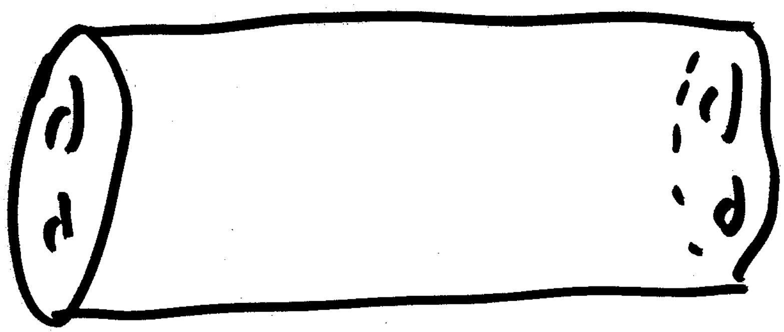
ASSOCIATED TO ρ IN THE REPRESENTATION ${}^L R$.

SO AFTER DUALITY, THE DUAL BRANE $\tilde{B}$



WILL BE AN EIGENBRANE FOR THE 't HOOFT OPERATORS WITH THE SAME "EIGENVALUE"

HOW DO 't HOOFT OPERATORS ACT ON BRANES?



$W = I \times C$

THEDERIVATION LEADS US TO NONLINEAR PDE's ASSOCIATED WITH SUPERSYMMETRY.

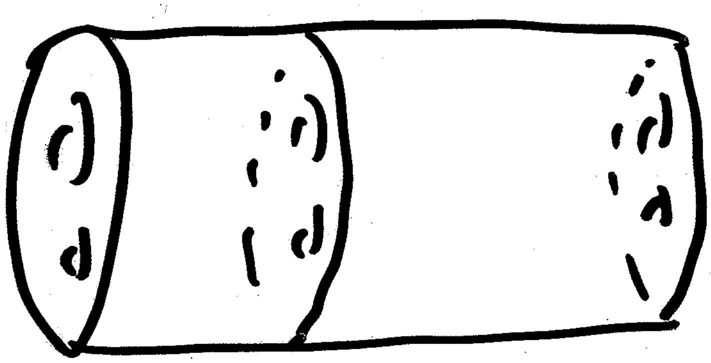
IF WE OMIT THE HIGGS FIELD, THESE BECOME EQNS FAMILIAR IN DONALDSON THEORY AND RELATED SUBJECTS

BOGOMOLNY EQNS IN THREE DIMENSIONS

$$F = * D\phi$$

FOR CONNECTION A ON G-BUNDLE $E \to W = $ 3-MANIFOLD

AND $\phi \in \Omega^0(W, ad(E))$



$W = I \times C$

RESTRICT TO

$$C_y = y \times C \qquad y \in I$$

ANY CONNECTION A

DETERMINES A HOLOMORPHIC BUNDLE

[redacted]

$$E_y = E\big|_{C_y}$$

$$E_y \to C_y$$

BOGOMOLNY EQNS [redacted]

IMPLY THAT THE HOLOMORPHIC

TYPE OF E_y IS INDEPENDENT

OF y

z = LOCAL COORD ON C

$F = dz\,d\bar{z}\,F_{y\bar{z}} + \dots$

$D\phi = d\bar{z}\,\frac{D\phi}{D\bar{z}} + \dots$

$$F_{y\bar{z}} = -i\,D_{\bar{z}}\,\phi$$

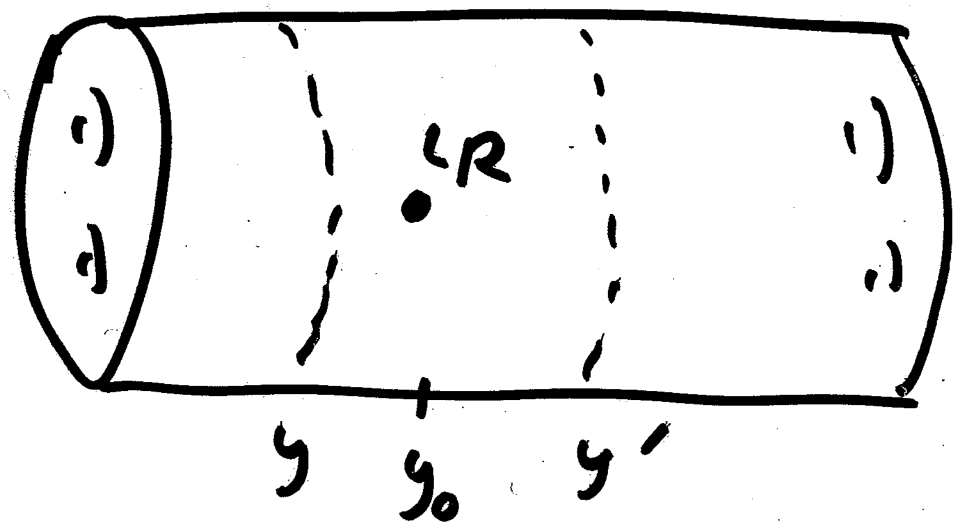
i.e.

$$\frac{\partial}{\partial y}\left(D_{\bar{z}}\right) = \left[D_{\bar{z}}, -i\phi\right]$$

THE $\bar{\partial}_A$ OPERATOR IS INDEPENDENT OF y UP TO CONJUGACY.

$$\bar{\partial}_A = d\bar{z}\ D_{\bar{z}}$$

NOW IF AN 't HOOFT OPERATOR IS PRESENT AT $y_0 \times p \in W$



FOR $y_0 \in I$, $p \in C$

THEN THE HOLOMORPHIC TYPE OF E_y JUMPS AT $y = y_0$.

$W \setminus (I \times p)$

BUT IT IS UNCHANGED

IF WE RESTRICT E_y

TO $C \setminus p$

THE CHANGE IN E_y IS

A "HECKE MODIFICATION"

OF TYPE $^L R$ ...

DETERMINED BY THE

'T HOOFT OPERATOR.

COISOTROPIC BRANE... KAP + ORLOR

~~CAN~~

ANY BRANE $\mathcal{B}_0$

$\mathcal{A} = (\mathcal{B}_0, \mathcal{B}_0)$ STRINGS FORM A RING

$\Rightarrow$ SHEAF OF RINGS

. . . .

FOR ANY $\mathcal{B}$

$(\mathcal{B}_0, \mathcal{B})$ STRINGS =

AN $\mathcal{A}$-MODULE

FIND "CANONICAL

COIS. BRANE $\mathcal{B}_0$"

s.t. $\mathcal{A}$ = { SHEAF OF

DIFF OPS

ON $\mathcal{M}$ }

$T^*\mathcal{M} \subset \mathcal{M}_H$