A

LET ℓ = 1-MANIFOLD

AND $\Phi : \ell \to X$

$$I = \frac{1}{2}\int dt \left|\frac{d\Phi}{dt}\right|^2$$

QUANTIZATION LEADS TO

FUNCTIONS ON X WITH

HAMILTONIAN = LAPLACIAN

THIS CASE, AND ITS SUPERSYMMETRIC

EXTENSIONS, ARE FAMILIAR

MATHEMATICALLY.

B

LESS FAMILIAR IS THE CASE IN WHICH $\mathbb{R}$ IS REPLACED BY A TWO-MANIFOLD Σ

$$\Phi : \Sigma \to X$$

$$I = \int_{\Sigma} \langle d\Phi, * d\Phi \rangle$$

= ACTION FOR HARMONIC MAP

QUANTUM THEORY:

FUNCTIONS ON FREE LOOP SPACE WITH APPROPRIATE ANALOG OF LAPLACIAN, etc.

C

THIS SECOND ONE IS MORE TYPICAL OF WHAT PHYSICISTS ACTUALLY DO AND IS THE NATURAL FRAMEWORK, FOR INSTANCE, FOR MIRROR SYMMETRY.

YESTERDAY WE STARTED WITH A FOUR-DIMENSIONAL GAUGE THEORY

$$I = \frac{1}{4e^2}\int \mathrm{Tr}\, F \wedge * F + \dots$$

AND "REDUCED" TO ONE DIMENSION SOLELY TO GET SOMETHING MORE FAMILIAR

D

HOWEVER, A DIFFERENT KIND OF REDUCTION IS ESSENTIAL FOR OUR PROBLEM:

LET $M = \Sigma \times C$ WITH A PRODUCT METRIC

MULTIPLY METRIC ON Σ BY REAL CONSTANT t AND TAKE t LARGE. TO KEEP ENERGY OR ACTION FROM DIVERGING, THE FIELDS, RESTRICTED TO $p \times C$ FOR $p \in \Sigma$

E

MUST OBEY CERTAIN MARVELOUS EQNS DUE TO HITCHIN

AS p VARIES, ONE GETS A SLOWLY VARYING MAP TO THE MODULI SPACE $\mathcal{M}_H$ OF HITCHIN'S EQUATIONS

EQNS TO MINIMIZE THE ENERGY ARE HITCHIN'S EQUATIONS

$$F - \phi \wedge \phi = 0$$

$$d_A \phi = d_A * \phi = 0$$

THE MODULI SPACE OF SOLUTIONS OF THIS EQUATION IS A HYPER-KAHLER MANIFOLD

$\mathcal{M}_H$ (OR $\mathcal{M}_H(G, C)$)

WITH MARVELOUS PROPERTIES

$\mathcal{N}=4$ SUPER YANG-MILLS THEORY ON $M = \Sigma \times C$ CAN BE USEFULLY APPROXIMATED, FOR OUR PURPOSES, BY A "SIGMA MODEL" OF MAPS

$$\Phi : \Sigma \to \mathcal{M}_H(G, C)$$

(ACTUALLY Σ SHOULD BE EXTENDED TO A SUPERMANIFOLD TO GIVE A SUPERSYMMETRIC MODEL)

BERSHADSKY, JOHANSEN, SADOV, VAFA (1995); HARVEY, MOORE, STROMINGER (1995)

THE FOUR-DIMENSIONAL ELECTRIC-MAGNETIC DUALITY

$$S : \tau \to -\frac{1}{n_g \tau}$$

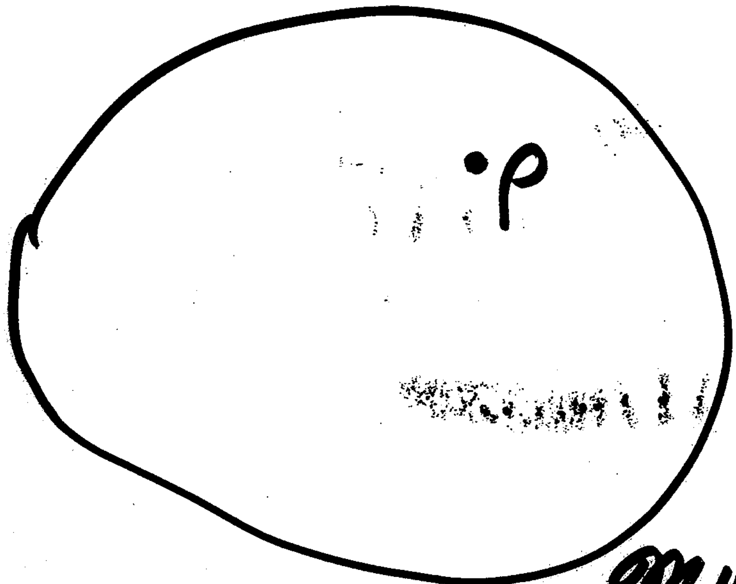
BECOMES A "MIRROR SYMMETRY" OF THE TWO-DIMENSIONAL SIGMA MODEL, AND THIS INSTANCE OF "MIRROR SYMMETRY" (HAUSEL & THADDEUS 2001) GIVES ESSENTIALLY, AFTER SOME WORK, THE USUAL STATEMENTS OF GEOMETRIC LANGLANDS....

B-MODEL OF $\mathcal{M}_H({}^LG, C)$ $\xleftrightarrow{\text{MIRROR SYMMETRY}}$ A-MODEL OF $\mathcal{M}_H(G, C)$

NOW START WITH

$$\rho : \pi_1(C) \to {}^LG_{\mathbb{C}}$$

ACCORDING TO HITCHIN, ρ DEFINES A POINT IN $\mathcal{M}_H({}^LG$



$\mathcal{M}_H({}^LG, C)$

A "ZERO-BRANE" SUPPORTED AT THIS POINT IS A BRANE OF THE B-MODEL OF $\mathcal{M}_H({}^LG, C)$

ITS MIRROR WILL BE AN A-BRANE ON $\mathcal{M}_H(G, C)$

$\mathcal{M}_H$ HAS SOME SPECIAL PROPERTIES, WHICH I'LL DESCRIBE [redacted], SUCH THAT A-BRANES ON $\mathcal{M}_H$ ARE NATURALLY ASSOCIATED TO "$\mathcal{D}$-MODULES" ON

$\mathcal{M}(G,C)$ = THE MODULI SPACE OF STABLE BUNDLES

MOREOVER, BY RETURNING TO FOUR DIMENSIONS AND CONSIDERING THE "WILSON AND 't HOOFT OPERATORS" OF THE UNDERLYING FOUR-DIMENSIONAL GAUGE THEORY, ONE CAN ARGUE THAT THE $\mathcal{D}$-MODULE DERIVED FROM $\rho : \pi_1(C) \to {}^L G$ IS A "HECKE EIGENSHEAF."

52

THREE LINEAR COMBINATIONS

OF THE THREE REAL
HITCHIN'S EQUATIONS
EQUATIONS COMBINE TO A

SINGLE HOLOMORPHIC

CONDITION. $\bar{\partial}_A \phi = 0$ THE THIRD IS

SOLUTION SPACE MAPIS CONDITION.

HYPER-KAHLER.
LET'S SEE HOW THIS WORKS

IN ANY COMPLEX STRUCTURE

I, J, K (OR $aI + bJ + cK$
USING HITCHIN'S NOTATION
$a^2 + b^2 + c^2 = 1$)

COMPLEX STRUCTURE J

THE HOLOMORPHIC EQUATION IS

$$F - \phi \wedge \phi + i\, d_A \phi = 0$$

OR $\quad \mathcal{F} = 0$.

WHERE

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

$$\mathcal{A} = A + i\phi$$

SO THE COMPLEX-VALUED CONNECTION $\mathcal{A}$ IS FLAT.

BY CORLETTE & DONALDSON

THE SPACE OF COMPLEX

FLAT CONNECTIONS WITH

VANISHING "MOMENT MAP"

$$D * \phi = 0$$

MOD G-VALUED GAUGE

TRANSFORMATIONS

IS THE MODULI SPACE OF

(STABLE) COMPLEX FLAT

CONNECTIONS MODULO $G_{\mathbb{C}}$-VALUED

GAUGE TRANSFORMATIONS.

SO $\mathcal{M}_H$ IN COMPLEX STRUCTURE J IS THE MODULI SPACE OF (STABLE) HOMOMORPHISMS

$$\rho : \pi_1(C) \to G_{\mathbb{C}}$$

A HOMOMORPHISM IS STABLE IF IT IS "IRREDUCIBLE" ... OTHERWISE SEMISTABLE (THE MODULI SPACE OF STABLE HOMOMORPHISMS INCLUDE POINTS REPRESENTING SEMI-STABLE EQUIVALENCE CLASSES.)

SO THIS EXPLAINS A STATEMENT I MADE BEFORE:

A HOMOMORPHISM

$$\rho : \pi_1(C) \to G_{\mathbb{C}}$$

DEFINES A POINT IN $\mathcal{M}_H(G, C)$.

COMPLEX STRUCTURE I

THE HOLOMORPHIC EQUATION

IS

$$d_A \phi + i * d_A \phi = 0$$

OR MORE SIMPLY

$$\bar{\partial}_A \varphi = 0$$

WHERE φ IS OF TYPE $(1,0)$

$$\phi = \varphi + \bar{\varphi}$$

THE MOMENT MAP EQN IS

$$F - \phi \wedge \phi = 0$$

THE HOLOMORPHIC EQN.

$$\bar{\partial}_A \varphi = 0$$

DEPENDS ON A ONLY VIA ITS $\bar{\partial}_A$ OPERATOR

$$\bar{\partial}_A = \bar{\partial} + A^{(0,1)}$$

i.e. ONLY VIA THE HOLOMORPHIC STRUCTURE WITH WHICH IT ENDOWS THE BUNDLE E.

HERE WE USE THE FACT THAT IN COMPLEX DIMENSION ONE, THERE IS NO OBSTRUCTION TO INTEGRABILITY, i.e.

$$(\bar\partial_A)^2 = 0 \quad \text{FOR ANY } A.$$

THE EQUATION

$$\bar\partial_A \varphi = 0$$

TELLS US THAT φ REPRESENTS AN ELEMENT OF

$$H^1(C, K_C \otimes ad(E))$$

K_C = CANONICAL BUNDLE OF C

SO (A, ϕ) WITH

$$\bar{\partial}_A \varphi = 0$$

DEFINE A "HIGGS BUNDLE"

i.e. A PAIR (E, φ)

$$\varphi \in H^1(C, K_C \otimes ad(E))$$

HITCHIN'S THEOREM IS THAT THE MOMENT MAP CONDITION

$$F - \phi \wedge \bar{\phi} = 0$$

PLUS THE OPERATION OF DIVIDING BY G-VALUED GAUGE TRANSFORMATIONS

GIVES US THE "MODULI

SPACE OF STABLE HIGGS

BUNDLES (E, φ)," UP

TO HOLOMORPHIC EQUIVALENCE

(i.e. $G_{\mathbb{C}}$ - VALUED GAUGE

TRANSFORMATIONS)

NOW IN COMPLEX STRUCTURE I, $\mathcal{M}_H$ HAS TWO MORE MARVELOUS PROPERTIES:

(a) LET E = A STABLE BUNDLE, REPRESENTING A POINT IN $\mathcal{M}$ = THE MODULI SPACE OF STABLE BUNDLES.

$H^1(C, K_C \otimes ad(E))$ IS THE FIBER AT E OF $T^*\mathcal{M}$ SO

(E, φ) DEFINES A POINT IN $T^*\mathcal{M}$

THIS CONSTRUCTION GIVES AN EMBEDDING OF $T^*\mathcal{M}$ AS A DENSE OPEN SET IN $\mathcal{M}_H$.

FOR MANY PURPOSES, ONE CAN APPROXIMATE $\mathcal{M}_H$ BY $T^*\mathcal{M}$.

(b) WITH RESPECT TO THE HOLOMORPHIC SYMPLECTIC STRUCTURE WHICH EXTENDS THAT OF THE COTANGENT BUNDLE $T^*\mathcal{M}$,

$\mathcal{M}_H$ IS A COMPLETELY INTEGRABLE HAMILTONIAN SYSTEM.

COMMUTING HAMILTONIANS FROM CHARACTERISTIC POLYNOMIAL OF φ.

FOR EXAMPLE, FOR $G = SU(2)$

$$\dim_{\mathbb{C}} \mathcal{M}_H = 6g - 6$$

$$g = \text{genus}(C)$$

WE NEED $3g-3$ COMMUTING HAMILTONIANS.

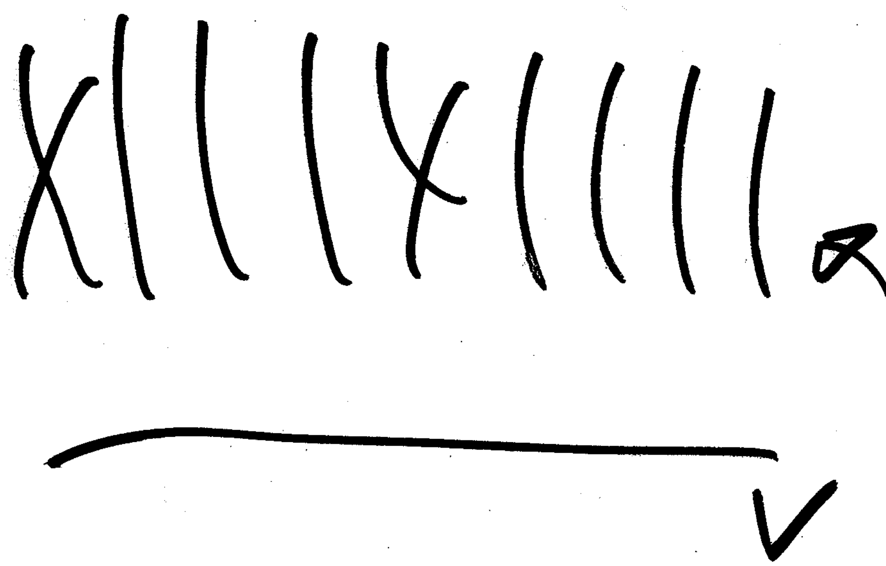
LET $V = H^0(C, K_C^2)$, OF DIMENSION $3g-3$

WE GET A MAP

$$\pi : \mathcal{M}_H \to V$$

BY $(E, \varphi) \to \mathrm{Tr}\, \varphi^2 \in V$

AND THIS MAP ESTABLISHES COMPLETE INTEGRABILITY



GENERIC FIBER = COMPLEX ABELIAN VARIETY

SINCE $\mathcal{M}_H$ IS HYPER-KAHLER,

IN EACH COMLEX STRUCTURE

I, J, K THERE IS A KAHLER FORM

$\omega_I, \omega_J, \omega_K$ AND A

HOLOMORPHIC TWO-FORM

$\Omega_I, \Omega_J, \Omega_K$.

WE HAVE $\Omega_I = \omega_J + i\omega_K$

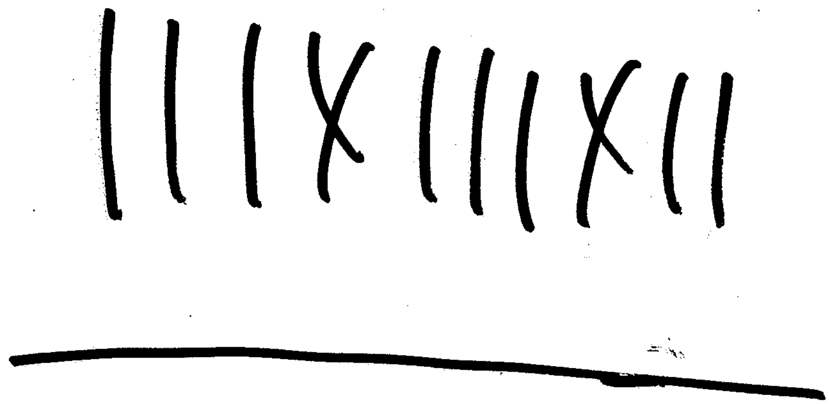
AND CYCLIC PERMUTATIONS.

Ω_I, RESTRICTED TO $T^*\mathcal{M} \subset \mathcal{M}_H$,

IS THE NATURAL HOLOMORPHIC

TWO-FORM OF $T^*\mathcal{M}$.

NOW OUR PICTURE



WITH GENERIC FIBER A TORUS

IS THE STROMINGER-YAU-ZASLOW

PICTURE OF MIRROR SYMMETRY

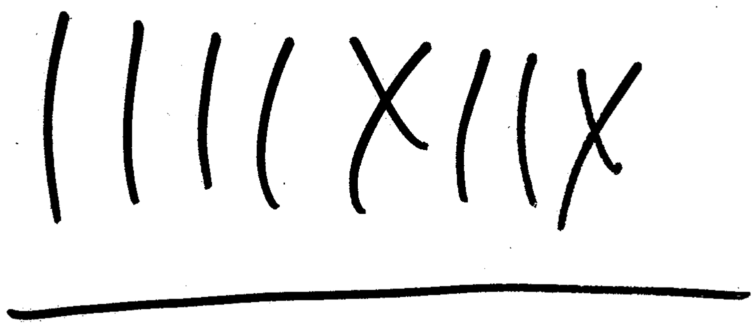
... BETWEEN THE B-MODEL IN

COMPLEX STRUCTURE J AND

THE A-MODEL OF

$\omega_K = \operatorname{Im} \Omega_I$

HOWEVER, IT IS DRASTICALLY SIMPLER THAN THE GENERIC CASE OF MIRROR SYMMETRY



BECAUSE $\mathcal{M}_H$ IS HYPER KAHLER AND THE FIBERS ARE HOLOMORPHIC IN COMPLEX STRUCTURE I.

THIS IS ACTUALLY A RARE CASE OF AN SYZ FIBRATION THAT CAN BE DESCRIBED VERY EXPLICITLY

(HAUSEL & THADDEUS)

THE UNDERLYING FOUR-DIMENSIONAL S-DUALITY $S: \tau \to -1/\tau$

REDUCES IN TWO DIMENSIONS TO THE MIRROR SYMMETRY

B-MODEL OF $\mathcal{M}_H({}^LG, C)$
IN COMPLEX STRUCTURE J

$\wr\wr$

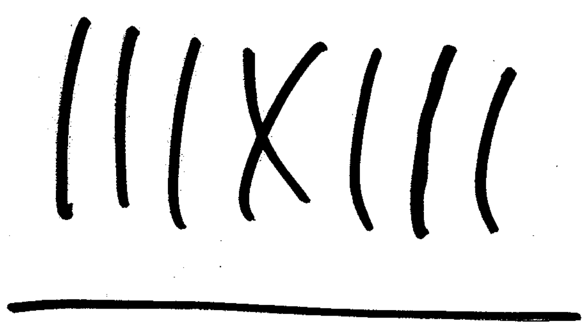
A-MODEL OF $\mathcal{M}_H(G, C)$
IN SYMPLECTIC STRUCTURE ω_K

SO $\rho : \pi_1(C) \to \mathcal{M}_H$

WHICH DETERMINES A B-BRANE

IS DUAL TO AN A-BRANE

OF SYZ TYPE



(FIBER ENDOWED WITH A

FLAT LINE BUNDLE)

MOREOVER, BECAUSE

$$\mathcal{M}_H \simeq T^*\mathcal{M}$$

THERE IS A NATURAL MAP

$\{$A-BRANES ON $\mathcal{M}_H\}$

TO

$\{$D-MODULES ON $\mathcal{M}\}$

SO THIS GIVES THE ASSOCIATION

$\{$FLAT ^LG BUNDLE $^LE \to C\}$

TO

$\{$D-MODULE ON $\mathcal{M}(G,C)\}$

OF THE GEOMETRIC LANGLANDS PROGRAM.

BUT I WANT TO EXPLAIN WHY THE A-BRANE CORRESPONDING TO A HOMOMORPHISM ρ IS ACTUALLY A "HECKE EIGENSHEAF"

FOR THIS, WE GO BACK TO FOUR DIMENSIONS.