Measurability Problems for Boolean Algebras

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Berkeley, March 31, 2014

Outline

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A **finitely additive measure** on a field $\mathcal{F}$ of subsets of some set X is a function

$$\mu : \mathcal{F} \to [0, \infty]$$

such that

1. $\mu(\emptyset) = 0$,
2. $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ for every finite sequence $A_1, ..., A_n$ of pairwise disjoint elements of $\mathcal{F}$.

μ is σ-**additive** if

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

for every sequence (A_n) of elements of $\mathcal{F}$.

General Massproblem:

Are there measures on a given field $\mathcal{F}$ of subsets of X that are

invariant relative a given group action on X, or are

strictly $\mathcal{I}$-positive relative to a given ideal $\mathcal{I} \subseteq \mathcal{F}$?

Remark

Note that the second version of the problems is really a problem about the existence of **strictly positive** measures on **Boolean algebras** $\mathbb{B}$ where in order to avoid trivialities we want **finite measures**, or more precisely, we want measures

$$\mu : \mathbb{B} \to [0, 1].$$

such that $\mu(1) = 1$.

Tarski's theorem

Theorem (Tarski, 1938)

Suppose a group G acts on a set X. Then the following are equivalent for a subset Y of X:

1. *There is a finitely additive G-invariant measure $\mu : \mathcal{P}(X) \to [0, \infty]$ such that $\mu(Y) = 1$.*
2. *There is no finite sequence $A_1, ..., A_m, B_1, ..., B_n$ of pairwise disjoint subsets of Y and a sequence $g_1, ..., g_m, h_1, .., g_n$ of elements of G such that*

$$\bigcup_1^m g_i(A_i) = \bigcup_1^n h_j(B_j) = Y.$$

Souslin's Massproblem

Problem (Souslin, 1920)

*The following are equivalent for every Boolean algebra $\mathbb{B}$ generated by a chain (***interval algebra***):*

1. $\mathbb{B}$ *supports a finitely additive strictly positive measure.*
2. *every family of pairwise disjoint elements of $\mathbb{B}$ is countable (***the countable chain condition***).*

Remark

Souslin originally asked whether every ordered continuum satisfying the **countable chain condition** is **separable**, or equivalently, **matrizable**.

Von Neumann's Massproblem

Problem (Von Neumann, 1937)

Suppose that a σ-complete Boolean algebra $\mathbb{B}$ has the following properties:

1. $\mathbb{B}$ *satisfies the* **countable chain condition**,
2. $\mathbb{B}$ *satisfies the* **weak countable distributive law**, *i.e.,*

$$\bigwedge_m \bigvee_n a_{mn} = \bigvee_f \bigwedge_m a_{mf(m)}$$

for sequences (a_{mn}) such that $a_{mn} \leq a_{mn'}$ when $n \leq n'$.

*Does $\mathbb{B}$ support a strictly positive σ-***additive measure***?*

Tarski's Massproblem

Problem (Tarski, 1939, 1945, 1948)

Suppose that a Boolean algebra $\mathbb{B}$ has one of the following properties

1. $\mathbb{B}$ *can be decomposed into a sequence* (S_n) *of subsets such that no* S_n *contain more than* n *pairwise disjoint elements* *(*σ*-***bounded chain condition***).*
2. $\mathbb{B}$ *can be decomposed into a sequence* (S_n) *of subsets none of which includes an infinite subset of pairwise disjoint elements* *(*σ*-***finite chain condition***).*
3. $\mathbb{B}$ *satisfies the* **countable chain condition***.*

Does $\mathbb{B}$ *support a strictly positive* **finitely additive measure***?*

Analysis of Tarski's Massproblem

A **functional** on $\mathbb{B}$ is simply a function

$$f : \mathbb{B} \to [0, \infty)$$

such that $f(0) = 0$.
A functiional $f : \mathbb{B} \to [0, \infty)$ is **exhaustive** whenever

$$\lim_{n\to\infty} f(a_n) = 0$$

for every sequence (a_m) of pairwise disjoint elements of $\mathbb{B}$.

Example

Outher measures are exhaustive

Proposition

*A Boolean algebra $\mathbb{B}$ satisfies σ-***finite chain condition** *if and only of it supports a strictly positive* **exhaustive functional***.*

A functiional

$$f : \mathbb{B} \to [0, \infty)$$

is **uniformly exhaustive** whenever for every $\varepsilon > 0$ there is integer $k(\varepsilon)$ such that

$$\min_{n < k(\varepsilon)} f(a_n) < \varepsilon$$

for every sequence $(a_n)_{n=1}^{k(\varepsilon)}$ of pairwise disjoint elements of $\mathbb{B}$.

Example

Inner measures are uniformly exhaustive.

Proposition

*A Boolean algebra $\mathbb{B}$ satisfies σ-***bounded chain condition** *if and only of it supports a strictly positive* **uniformly exhaustive functional***.*

Analysis of Von Neumann's Massproblem

A **submeasure** on a Boolean algebra $\mathbb{B}$ is function

$$\nu : \mathbb{B} \to [0, 1]$$

such that

1. $\nu(0) = 0$ and $\nu(1) = 1$,
2. $\nu(a) \leq \nu(b)$ whenever $a \leq b$,
3. $\nu(a \vee b) \leq \nu(a) + \nu(b)$.

ν is **strictly positive** on $\mathbb{B}$ if

$$\nu(a) > 0 \text{ for } a \neq 0.$$

ν is **continuous** whenever

$$a_n \downarrow 0 \text{ implies } \nu(a_n) \downarrow 0.$$

Topology of sequential convergence

The **topology of sequential convergence** on a σ-complete Boolean algebra $\mathbb{B}$ is the **largest topology** on $\mathbb{B}$ in which **algebraically convergent** sequences are **convergent**, where

$$a_n \to a \text{ iff } a = \bigvee_m \bigwedge_{n \geq m} a_n = \bigwedge_m \bigvee_{n \geq m} a_n.$$

For a given subset A of $\mathbb{B}$, let

$$\overline{A} = \{a \in \mathbb{B} : (\exists (a_n) \subseteq A) \;\; a_n \to a\}$$

Proposition (Maharam, 1947)

Suppose $\mathbb{B}$ is a complete Boolean algebra satisfying the **countable chain condition** *and the* **weak countable distributive law**. *Then $A \mapsto \overline{A}$ is a closure operator giving the topology of sequential convergence on $\mathbb{B}$.*

Theorem (Balcar, Glowczynski, Jech, 1998)

The following conditions are equivalent for a complete Boolean algebra $\mathbb{B}$ satisfying the countable chain condition:

1. *The topology of sequential convergence on $\mathbb{B}$ is* **Hausdorff***.*
2. *The topology of sequential convergence on $\mathbb{B}$ is* **metrizable***.*

Theorem (Maharam, 1947)

The following are equivalent for a σ-complete Boolean algebra $\mathbb{B}$:

1. *$\mathbb{B}$ supports a strictly positive* **continuous submeasure***.*
2. *The topology of sequential convergence on $\mathbb{B}$ is* **metrizable***.*

Two Massproblems of Maharam

Problem (Maharam, 1947)

Suppose that a σ-complete Boolean algebra $\mathbb{B}$ has the following properties:

1. $\mathbb{B}$ *satisfies the* **countable chain condition**,
2. $\mathbb{B}$ *satisfies the* **weak countable distributive law**.

Does $\mathbb{B}$ *support a strictly positive* **continuous submeasure**?

Problem (Maharam, 1947)

If a σ-complete Boolean algebra supports a strictly positive **continuous submeasure** *does it also support a strictly positive* σ-**additive measure**?

Remark

The second problem reappeared in some areas of functional analysis as the **Control Measure Problem**.

The Control Measure Problem

A function $\nu : \mathbb{B} \to [0,1]$ is **exhaustive** if for every $\varepsilon > 0$ the set

$$\{a \in \mathbb{B} : \nu(a) \geq \varepsilon\}$$

contains no **infinite** sequence (a_n) such that $a_m \wedge a_n = 0$ for $m \neq n$.

Example

Continuous submeasures are exhaustive

A function $\nu : \mathbb{B} \to [0,1]$ is **uniformly exhaustive** if for every $\varepsilon > 0$ there is $k(\varepsilon) \in \mathbb{N}$ such that the set

$$\{a \in \mathbb{B} : \nu(a) \geq \varepsilon\}$$

contains no sequence $(a_n)_{n=1}^{k(\varepsilon)}$ such that $a_m \wedge a_n = 0$ for $m \neq n$.

Example

Finitely additive measures are uniformly exhaustive

A submeasure ν_1 is **absolutely continuous** relative to a submeasure ν_2 whenever

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall a \in \mathbb{B})[\nu_2(a) < \delta \to \nu_1(a) < \varepsilon].$$

Problem

Is every **exhaustive submeasure** *absolutely continuous relative to a* **finitely additive measure***?*

Theorem (Kalton-Roberts, 1983)

Every **uniformly exhaustive submeasure** *is absolutely continuous relative to a finitely additive* **measure***.*

Problem

Is every exhaustive submeasure uniformly exhaustive?

Let X be a **metrizable topological vector space** and d be a translation invariant metric on X that defines the topology.

A **vector messure** on a Boolean Algebra $\mathbb{B}$ is a map

$$\tau : \mathbb{B} \to X$$

such that

1. $\tau(0) = 0$, and
2. $\tau(a \vee b) = \tau(a) + \tau(b)$ whenever $a \wedge b = 0$.

We say that a vector measure $\tau : \mathbb{B} \to X$ is **exhaustive** whenever

$$\lim_{n \to \infty} \tau(a_n) = 0$$

for all sequences (a_n) of pairwise disjoint elements of $\mathbb{B}$.
A finitely additive measure $\mu : \mathbb{B} \to [0, 1]$ is a **control measure** of a vector measure $\tau : \mathbb{B} \to X$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall a \in \mathbb{B})[\mu(a) < \delta \to d(0, \tau(a)) < \varepsilon].$$

Problem

Is every vector measure on a given Boolean algebra $\mathbb{B}$ controlled by a finitely additive measure on $\mathbb{B}$?

Theorem (Talagrand, 2006)

There is an exhaustive measure on the countable free Boolean algebra which is not uniformly exhaustive.

Corollary (Talagrand, 2006)

There is a σ-complete Boolean algebra which supports a strictly positive continuous submeasure but it does not support any strictly positive measure.

The Horn-Tarski Problem

Problem (Horn and Tarski, 1948)

Are the following two conditions equivalent for a given Boolean algebra $\mathcal{B}$?

1. $\mathbb{B}$ *supports a strictly positive* **exhaustive** *function* $\nu : \mathbb{B} \to [0, 1]$.
2. $\mathbb{B}$ *supports a strictly positive* **uniformly exhaustive** *function* $\nu : \mathbb{B} \to [0, 1]$.

Equivalently, are the following conditions equivalent for a given Boolean algebra $\mathcal{B}$?

1. $\mathbb{B}$ *satisfies the* σ-**finite chain condition**.
2. $\mathbb{B}$ *satisfies the* σ-**bounded chain condition**.

Kelley's criterion and Gaifman's example

Theorem (Kelley, 1959)

A Boolean algebra supports a strictly positive finitely additive measure if and only if it can be decomposed into a sequence of subsets having **positive intersection numbers***.*

The **intersection number** of a subset S of $\mathbb{B} \setminus \{0\}$ is defined as

$$I(S) = \inf\{\frac{|X|}{|Y|} : X, Y \in [S]^{<\omega}, X \subseteq Y, \bigwedge X \neq 0\}.$$

Theorem (Gaifman, 1963)

There is a Boolean algebra $\mathbb{B}$ satisfying the **σ-bounded chain condition** *which* **does not support** *a strictly positive finitely additive* **measure***.*

An algebraic characterization

Theorem (T., 2004)

A σ-complete Boolean algebra $\mathbb{B}$ supports a strictly positive **continuous submesure** *if and only if,*

1. $\mathbb{B}$ *satisfies the* σ-**finite chain condition**, *and*
2. $\mathbb{B}$ *satisfies the* **weak countable distributive law**.

Remark

The proof shows that we can weaken the σ-**finite chain condition** to the **effectively provable countable chain condition**. We shall explore this idea below.

Borel theory

The field of **Borel subsets** of a topological space X is the smallest σ-field of subsets of X that contains all open subsets of X.

A **base** of a Boolean algebra $\mathbb{B}$ is any set $\mathbb{P} \subseteq \mathbb{B}^+ = \mathbb{B} \setminus \{0\}$ such that for every $b \in \mathbb{B}^+$ we can find $a \in \mathbb{P}$ such that $a \leq b$.

A base $\mathbb{P}$ is **Borel** if it can be represented as a **Borel structure** on some of the standard spaces such as $\mathbb{R}$, $[0,1]$, $\{0,1\}^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, etc.

Theorem (T., 2004)

A σ-complete Boolean algebra $\mathbb{B}$ with a Borel base supports a strictly positive **continuous submeasure** *if and only if*

1. $\mathbb{B}$ *satisfies the* **countable chain condition**, *and*
2. $\mathbb{B}$ *satisfies the* **weak countable distributive law**.

The functor $\mathbb{T}$

Fix a topological space X. For $D \subseteq X$, let

$$D^{(1)} = \text{the set of non-isolated points of D.}$$

Let $\mathbb{T}(X)$ be the collection of all partial functions

$$p : X \to 2$$

such that:

1. $|dom(p)| \leq \aleph_0$,
2. $|p^{-1}(1)| < \aleph_0$,
3. $(dom(p))^{(1)} \subseteq p^{-1}(1)$.

The ordering on $\mathbb{T}(X)$ is the reverse inclusion.

$\mathbb{T}(X)$ is a base of a complete Boolean algebra denoted by $\mathbb{B}(X)$.

Theorem (T., 1991)

1. *If X is a metric space $\mathbb{T}(X)$ satisfies the* **countable chain condition**.
2. *If X is a Polish space, $\mathbb{T}(X)$ is a* **Borel partially ordered set**.
3. *If X is a Polish space with no isolated points, $\mathbb{T}(X)$* **fails** *to satisfy the σ-***finite chain condition**.
4. $\mathbb{B}(X)$ **does not support** *a strictly positive finitely additive* **measure** *unless X is countable.*

Theorem (Balcal, Pazák, Thümmel, 2012)

1. *X and Y are homeomorphic iff $\mathbb{T}(X)$ and $\mathbb{T}(Y)$ are isomorphic.*
2. *$\mathbb{B}([0,1])$ and $\mathbb{B}((0,1))$ are isomorphic.*
3. *$\mathbb{B}([0,1])$ is homogeneous.*

Problem

Are $\mathbb{B}([0,1])$ and $\mathbb{B}([0,1] \times [0,1])$ isomorphic?

Two solutions to the Horn-Tarski problem

Theorem (Thümmel, 2012)

There is a first countable topological space X such that

1. $\mathbb{T}(X)$ *satisfies the* σ-**finite chain condition**, *but*
2. $\mathbb{T}(X)$ *does* **not** *satisfy the* σ-**bounded chain condition**.

Theorem (T., 2013)

There is a **Borel** *first contable topological space X such that*

1. $\mathbb{T}(X)$ *is a* **Borel partially ordered set**,
2. $\mathbb{T}(X)$ *satisfies the* σ-**finite chain condition**, *but*
3. $\mathbb{T}(X)$ *does* **not** *satisfy the* σ-**bounded chain condition**.

Control Submeasure Problem

Problem

Suppose that a complete Boolean algebra $\mathbb{B}$ supports a strictly positive **continuous submeasure***.*
*Does $\mathbb{B}$ satisfy the σ-***bounded chain condition***?*

Equivalently, suppose that a Boolean algebra $\mathbb{B}$ supports a strictly positive **exhaustive submeasure**

$$\nu : \mathbb{B} \to [0, 1].$$

Does $\mathbb{B}$ necessarily support a strictly positive **uniformly exhaustive functional**

$$f : \mathbb{B} \to [0, 1]?$$

P-Ideal Dichotomy and the Massproblem

Definition

An ideal $\mathcal{I}$ of countable subsets of some set S is a **P-ideal** if for every sequence $(a_n) \subseteq \mathcal{I}$ there is $b \in \mathcal{I}$ such that

$$a_n \setminus b \text{ is finite for all } n.$$

Definition

The **P-ideal dichotomy** is the statement that for every P-ideal $\mathcal{I}$ of countable subsets of some index-set S, either

1. there is uncountable set $T \subseteq S$ such that $\mathcal{I}$ includes **all** countable subsets of T, or
2. S can be decomposed into a sequence (S_n) of subsets such that no S_n contains an infinite subset belonging to $\mathcal{I}$.

Theorem (T., 1985, 2000)

The P-Ideal Dichotomy is consistent relative to the consistency of a supercompact cardinal.
Moreover, PID is consistent with GCH.

Theorem (T., 2000)

PID imples $\mathfrak{b} \leq \aleph_2$.

Theorem (T., 2000)

PID implies that $\square_\kappa$ *fails for all cardinals* $\kappa \geq \omega_1$.

Theorem (Viale 2008)

PID implies the Singular Cardinals Hypothesis.

Problem

Does PID imply $2^{\aleph_0} \leq \aleph_2$*?*

P-ideal of converging sequences

For a σ-complete Boolean algebra $\mathbb{B}$, let

$$\mathcal{I}_{\mathbb{B}} = \{(a_n) \subseteq \mathbb{B} \setminus \{0\} : a_n \to 0\}.$$

Then

1. $\mathcal{I}_{\mathbb{B}}$ is a **P-ideal** iff $\mathbb{B}$ satisfies the **weak distributive law**.
2. If $\mathbb{B}$ satisfies the **countable chain condition** then the **first alternative of PID is false,** i.e., there is no uncountable $T \subseteq \mathbb{B}$ such that $\mathcal{I}_{\mathbb{B}}$ includes all countable subsets of T.

Theorem (Maharam 1947; Balcar, Jech, Pazák, 2003)

The following conditions are equivalent for a complete Boolean algebra $\mathbb{B}$:

1. $\mathbb{B}$ *supports a strictly positive* **continuous submeasure**.
2. *The topology of sequential convergence on* $\mathbb{B}$ *is* **metrizable**.
3. 0 **is a** G_δ**-point** *relative the topology of sequential convergence on* $\mathbb{B}$.

Theorem (Balcar, Jech, Pazák, 2003)

Assume PID. A σ*-complete Boolean algebra* $\mathbb{B}$ *supports a strictly positive* **continuous submeasure** *if and only if*

1. $\mathbb{B}$ *satisfies the* **countable chain condition**, *and*
2. $\mathbb{B}$ *satisfies the* **weak countable distributive law**.