RECIPROCITY LAWS AND DENSITY THEOREMS

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General problem: count the number of solutions to a FIXED polynomial(s) modulo a VARIABLE PRIME number.

RECIPROCITY LAW: a law which gives a completely different way to find the number of solutions for any given prime p.

DENSITY THEOREM: a theorem which describes the statistical behaviour of the number of solutions as the prime p varies.

GAUSS' LAW OF QUADRATIC RECIPROCITY (1796):

For any whole number n and prime number p the number of solutions to

$X^2 \equiv n$ **modulo** p

is 0, 1 **or** 2. **For fixed** n **it depends only on** p **modulo** $4n$.

How many solutions does $X^2 + 7 \equiv 0$ **have modulo** 32452843**?**

$32452843 = 1159030 \times 28 + 3$

Thus it has the same number of solutions as does

$X^2 + 7 \equiv 0$ **modulo** 3**,**

i.e. none.

DISTRIBUTION QUESTIONS

For what fraction of prime numbers p does $X^2 + n \equiv 0$ modulo p have 2 solutions? And what fraction 0 solutions?

THEOREM (Dirichlet, 1837): If $-n$ is not a perfect square then for half the primes $X^2 + n \equiv 0$ modulo p has two solutions and for half the primes it has no solutions.

More precisely de la Vallée-Poussin showed in 1896 that

$$\frac{\#\{p \le t: \ X^2 + n \equiv 0 \bmod p \text{ has no solutions}\}}{\#\{p \le t\}}$$

and

$$\frac{\#\{p \le t: \ X^2 + n \equiv 0 \bmod p \text{ has two solutions}\}}{\#\{p \le t\}}$$

(where p denotes a variable prime number) both tend to $1/2$ as t tends to infinity.

Both Dirichlet and de la Vallée-Poussin used Gauss' law of quadratic reciprocity in an essential way.

What about higher degree polynomials of one variable?

There is a reciprocity theorem conjectured by Langlands, but it still seems to be far from being proved. It is not known even for a general quintic equation.

However, rather surprisingly, Dirichlet's density theorem was extended to ALL one variable polynomial equations by Frobenius in 1880.

Example:

$$X^4 - 2 = 0.$$

Its GALOIS GROUP G consists of all permutations of the roots

$$\{\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}\}$$

which preserve all algebraic relations between them. For instance

$$\sqrt[4]{2} + (-\sqrt[4]{2}) = 0$$

and so the pair $\{\sqrt[4]{2}, -\sqrt[4]{2}\}$ must be taken either to itself or to the pair $\{i\sqrt[4]{2}, -i\sqrt[4]{2}\}$.

$$1$$

$$(\sqrt[4]{2},\ i\sqrt[4]{2},\ -\sqrt[4]{2},\ -i\sqrt[4]{2})$$

$$(\sqrt[4]{2},\ -\sqrt[4]{2})(i\sqrt[4]{2},\ -i\sqrt[4]{2})$$

$$(\sqrt[4]{2},\ -i\sqrt[4]{2},\ -\sqrt[4]{2},\ i\sqrt[4]{2})$$

$$c = (i\sqrt[4]{2},\ -i\sqrt[4]{2})$$

$$(\sqrt[4]{2},\ -i\sqrt[4]{2})(-\sqrt[4]{2},\ i\sqrt[4]{2})$$

$$(\sqrt[4]{2},\ -\sqrt[4]{2})$$

$$(\sqrt[4]{2},\ i\sqrt[4]{2})(-\sqrt[4]{2},\ -i\sqrt[4]{2})$$

There are 8 such permutations:

1 fixes all four roots;

2 fix just two roots; and

5 fix no roots.

Frobenius and de la Vallée-Poussin showed that

$$\frac{\#\{p \le t :\ X^4 - 2 \equiv 0 \text{ mod } p \text{ has 0 solutions}\}}{\#\{p \le t\}} \longrightarrow 5/8$$

$$\frac{\#\{p \le t :\ X^4 - 2 \equiv 0 \text{ mod } p \text{ has 1 solution}\}}{\#\{p \le t\}} \longrightarrow 0$$

$$\frac{\#\{p \le t :\ X^4 - 2 \equiv 0 \text{ mod } p \text{ has 2 solutions}\}}{\#\{p \le t\}} \longrightarrow 1/4$$

$$\frac{\#\{p \le t :\ X^4 - 2 \equiv 0 \text{ mod } p \text{ has 3 solutions}\}}{\#\{p \le t\}} \longrightarrow 0$$

$$\frac{\#\{p \le t :\ X^4 - 2 \equiv 0 \text{ mod } p \text{ has 4 solutions}\}}{\#\{p \le t\}} \longrightarrow 1/8$$

as t goes to infinity.

What about equations with more variables?

For example (elliptic curves):

$Y^2 = X^3 + cX + d$

(c, d being fixed integers. Smooth, i.e. $4c^3 + 27d^2 \neq 0$.)

$j_E = 6912c^3/(4c^3 + 27d^2)$ **is the j-invariant of E.**

How does the number N_p of solutions modulo p vary with a prime number p?

$$E_0 : Y^2 + Y = X^3 - X^2$$

p	2	3	5	7	11	13	17	19	...
$p - N_p$	-2	-1	1	-2	1	4	-2	0	...

$$Y^2 + Y = X^3 - X^2$$

p	2	3	5	7	11	13	17	19	...
$p - N_p$	-2	-1	1	-2	1	4	-2	0	...

$$q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2 =$$

$$q - \mathbf{2q^2} - \mathbf{q^3} + 2q^4 + \mathbf{q^5} + 2q^6 - \mathbf{2q^7}$$

$$-2q^9 - 2q^{10} + \mathbf{q^{11}} - 2q^{12} + \mathbf{4q^{13}} + 4q^{14}$$

$$-q^{15} - 4q^{16} - \mathbf{2q^{17}} + 4q^{18} + 2q^{20} + ...$$

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$$-q^{15} - 4q^{16} - \mathbf{2q^{17}} + 4q^{18} + 2q^{20} + \ldots$$

THEOREM (Eichler, 1954)

$p - N_p$ **is the coefficient of** q^p**.**

$$f(z) = e^{2\pi iz}\prod_{n=1}^{\infty}(1-e^{2n\pi iz})^2(1-e^{22n\pi iz})^2 = \sum_{n=1}^{\infty} a_n e^{2n\pi iz}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z})$ **with** $11|c$ **implies**

$$f((az+b)/(cz+d)) = (cz+d)^2 f(z)$$

Also

$$f(-1/(11z)) = -11z^2 f(z)$$

TANIYAMA('55)-SHIMURA('57)-WEIL('67) CONJECTURE: Gives a somewhat similar effective algorithm for calculating $p - N_p$ for any elliptic curve

$E: \; Y^2 = X^3 + cX + d$ **(smooth).**

Proved (Breuil, Conrad, Diamond, T: 2001) following ideas introduced by Wiles.

The algorithm involves finite index subgroups of $GL_2(\mathbf{Z})$ the group of 2×2 matrices with whole number entries and determinant ± 1 and its action on the hyperbolic plane.

LANGLANDS in the mid 1970's proposed a similar reciprocity law for any system of polynomial equations in any number of variables in terms connected to subgroups of finite index in $GL_n(\mathrm{Z})$ for variable n.

We are beginning to make progress. For example Tom Barnet-Lamb (2009) has proved a reciprocity for

$$X_1^5+X_2^5+X_3^5+X_4^5+X_5^5 = aX_1X_2X_3X_4X_5$$

for $a \in \mathbf{Q}-\mathbf{Z}[1/10]$ **in terms of** $GL_4(\mathbf{Z})$ **and** $GL_2(\mathbf{Z})$**. He deduces the meromorphic continuation and functional equation of the** ζ**-function.**

DENSITY THEOREMS IN > 1 VARIABLE

$$E : Y^2 = X^3 + cX + d$$

THEOREM (Hasse, 1933): $|p - N_p| < 2\sqrt{p}$.

QUESTION: How is the normalised error term $(p - N_p)/\sqrt{p}$ distributed as p varies?

CONJECTURE (Sato-Tate, 1963): **If** E **is not CM then** $(p - N_p)/\sqrt{p}$ **is distributed in the range from** -2 **to** 2 **like**

$$(1/2\pi)\sqrt{4 - t^2}\,dt.$$

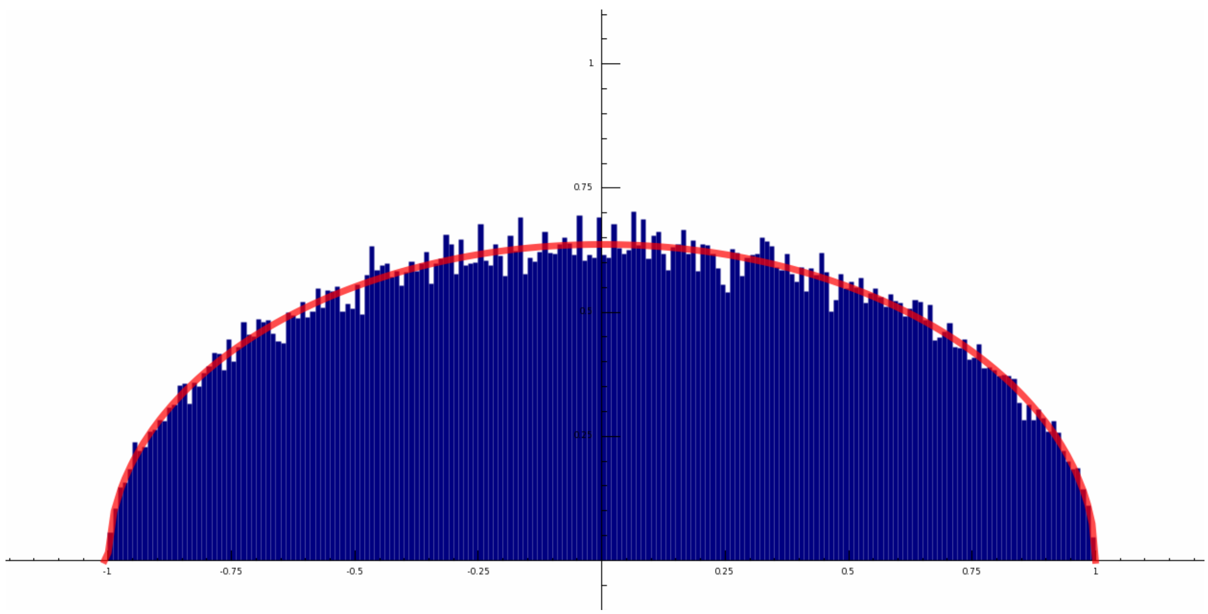
i.e. for $f \in C[-2, 2]$

$$\#\{p \le x\}^{-1} \sum_{p \le x} f((p - N_p)/\sqrt{p})$$

tends to

$$(1/2\pi)\int_{-2}^{2} f(t)\sqrt{4 - t^2}\,dt$$

as $x \to \infty$**.**



SATO-TATE DISTRIBUTION
FOR Δ AND p <1,000,000

(drawn by WILLIAM STEIN)

THEOREM (CHSBT, 2006): True if $j_E \in \mathbf{Q} - \mathbf{Z}$.

There exist conjectural generalizations to any number of polynomial equations in any number of variables.

$$
\begin{aligned}
SU(2)/\text{conjugacy} &\xrightarrow{\sim} [-2,2] \\
[g] &\longmapsto \operatorname{tr} g \\
\text{Haar measure} &\longleftrightarrow (1/2\pi)\sqrt{4-t^2}\,dt \\
[F_p/\sqrt{p}] &\longmapsto (p-N_p)/\sqrt{p},
\end{aligned}
$$

where $[F_p] \subset GL_2(\overline{\mathbf{Q}})$ **has characteristic polynomial**

$$X^2 - (p - N_p)X + p.$$

(Frobenius conjugacy class.)

The Sato-Tate conjecture says that the conjugacy classes

$$[F_p/\sqrt{p}]$$

are equidistributed in $SU(2)$/conjugacy **with respect to Haar measure.**

We have to prove that for all $f \in C[-2,2]$

$$\left(\sum_{p\le x} f(\mathrm{tr}\, F_p/\sqrt{p})\right)/\#\{p \le x\}$$

tends to

$$(1/2\pi)\int_{-2}^{2} f(t)\sqrt{4-t^2}\,dt$$

as $x \to \infty$**.**

The Peter-Weyl theorem tells us that a the functions

$$\mathrm{tr}\,\mathrm{Sym}^{\,n-1}$$

for $n = 1, 2, 3, \ldots$ **span a dense subspace of** $C[SU(2)/\text{conjugacy}] = C[-2, 2]$**.**

Hence it suffices to show that

$$\left(\sum_{p \le x} \mathrm{tr}\,\mathrm{Sym}^{\,n-1}(F_p/\sqrt{p})\right) /\#\{p \le x\}$$

tends to 1 **if** $n = 1$ **(clear) and tends to** 0 **if** $n > 1$**.**

L-FUNCTIONS: We define a holomorphic function

$$L(\mathsf{Symm}^{\,n-1}E, s)$$

in $\mathsf{Re}\, s > (n+1)/2$ **by**

$$\prod_p \mathsf{det}\left(1_n - (\mathsf{Symm}^{\,n-1}F_p)/p^s\right)^{-1}.$$

e.g.

$L(\mathsf{Symm}^{\,0}E, s) = \zeta(s)$

$L(\mathsf{Symm}^{\,1}E, s) = L(E, s)$

Taking logarithmic differentials we see that

$$L'(\mathrm{Symm}^{n-1}E,s)/L(\mathrm{Symm}^{n-1}E,s)$$

differs from

$$-\sum_p (\log p)(\mathrm{tr}\,\mathrm{Symm}^{n-1}(F_p/\sqrt{p}))p^{(n-1)/2-s}$$

by a function holomorphic in Re $s > n/2$**.**

Tauberian theorems tell us it suffices that the ratio is holomorphic in Re$s \geq (n+1)/2$**.**

i.e. that

$$L(\text{Symm}^{\,n-1}E, s)$$

is holomorphic and non-zero in

$$\text{Re}s \geq (n+1)/2$$

for $n > 1$**.**

Gelbart-Jacquet (1972): this is true IF $\text{Symm}^{\,n-1}E$ **satisfies a reciprocity law involving** $GL_n(\mathbf{Z})$**.**