UC Berkeley Math 10B, Spring 2015: Midterm 2

Prof. Sturmfels, April 9, **SOLUTIONS**

1. (5 points) You are a pollster for the 2016 presidential elections. You ask 1000 random people whether they would vote for the Republican candidate or the Democratic candidate. 560 people say that they will vote for the Democrat.

(a) Find a 95% confidence interval for the percentage of votes p that the Democrat will get. (You do not need to simplify your answer).

Let p be the random variable which measures the percentage vote that the Democratic candidate gets. We assume that the polling process follows a Binomial distribution, i.e. each voter's preference is independent of other voters.

The sample gives us the observed percentage, $\hat{p} = \frac{560}{1000} = \frac{56}{100}$. Assuming binomial distribution, the sample standard error is $\hat{SE} = \sqrt{\hat{p}(1-\hat{p})} = \sqrt{\frac{56}{100}\left(1 - \frac{56}{100}\right)}$

The 95% confidence interval is given by the following:

$$\left(\hat{p} - \frac{2\hat{SE}}{\sqrt{n}}\ ,\ \hat{p} + \frac{2\hat{SE}}{\sqrt{n}}\right).$$

Substituting the values above, we find

$$\boxed{\left(\frac{56}{100} - 2\sqrt{\frac{\frac{56}{100}\left(1 - \frac{56}{100}\right)}{1000}}\ ,\ \frac{56}{100} + 2\sqrt{\frac{\frac{56}{100}\left(1 - \frac{56}{100}\right)}{1000}}\right)} \approx (0.5286, 0.5914).$$

(b) Based on the information you have, if you wanted to be 99% confident instead of 95%, will you have to increase the size of your confidence interval from part(a) or decrease it? Explain briefly in your own words.

If I want to increase my confidence, I must opt for $\boxed{\text{a bigger interval}}$ because then I can be more sure that the parameter I am looking for is included in it.

2. (5 points) Find an explicit formula for the sequence $\{a_n\}_{n\geq 0}$ which satisfies $a_0 = 1$, $a_1 = 4,\ a_2 = 4$, and

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \qquad \text{for } n \geq 3.$$

We rewrite the given recurrence relation as

$$a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0 \tag{1}$$

from which we deduce that the characteristic polynomial is x^3-2x^2-x+2. Inspection shows that $r = +1$ and $r = -1$ are roots. Since the product of all three roots is the constant term $+2$, the third root is $r = -2$. So, the characteristic polynomial this factors as $(x-1)(x+1)(x-2)$, and the general solution to the recurrence is

$$a_n \ = \ c_1 \cdot 1^n \ + \ c_2 \cdot (-1)^n \ + \ c_3 \cdot 2^n \quad \text{for } n \geq 0. \tag{2}$$

We now use the given initial conditions to determine c_1, c_2, c_3. Substituting $n = 0, 1, 2$ into (2) and applying the initial conditions gives the system of linear equations

$$c_1 + c_2 + c_3 = 1 \tag{3}$$

$$c_1 - c_2 + 2c_3 = 4 \tag{4}$$

$$c_1 + c_2 + 4c_3 = 4 \tag{5}$$

in the variables c_1, c_2, c_3. Subtracting (5) from (3) gives $3c_3 = 3$; so $c_3 = 1$. Substituting into (3) and (4) gives

$$c_1 + c_2 = 0 \tag{6}$$

$$c_1 - c_2 = 2 \tag{7}$$

from which we obtain $c_1 = 1$ and $c_2 = -1$. Hence the desired formula is

$$\boxed{a_n \ = \ 1 \ - \ (-1)^n \ + \ 2^n} \qquad \text{for } \ n \geq 0.$$

3. (5 points) Is the function

$$f(x) = \sin x$$

a solution to the differential equation:

$$f'''(x) = f''(x) + f(x) \qquad \text{for all} \quad -\infty < x < \infty\,?$$

Why or why not? Give an explanation in complete sentences.

We plug in $f(x) = \sin x$ into both sides of the differential equation. The left hand side evaluates to the function

$$f'''(x) \;=\; (\cos x)'' \;=\; (-\sin x)' \;=\; -\cos x.$$

The right hand side evaluates to zero:

$$f''(x) + f(x) \;=\; (\cos x)' + \sin x \;=\; (-\sin x) + \sin x \;=\; 0.$$

Since it is not true that $-\cos x$ equals 0 for all $-\infty < x < \infty$, we conclude that the given function $f(x) = \sin x$ is $\boxed{\text{not a solution}}$ to the differential equation.

4. (5 points) Is the sequence defined by

$$a_n = \frac{1}{2}n(n+1) \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

a solution to the recurrence relation:

$$a_n = 2a_{n-1} - a_{n-2} + 1 \quad \text{for all } \ n = 2, 3, 4, \ldots?$$

Why or why not? Give an explanation in complete sentences.

We plug in $a_n = \frac{1}{2}n(n+1)$ for $n = 0, 1, 2, \ldots$ into the recurrence relation. In order to do this, we need to plug in $n-1$ and $n-2$ for n in $a_n = \frac{1}{2}n(n+1)$ to get:

$$a_{n-1} = \frac{1}{2}(n-1)n, \qquad a_{n-2} = \frac{1}{2}(n-2)(n-1)$$

for all $n = 2, 3, 4, \ldots$.

For all integers $n = 2, 3, 4, \ldots$, the following equations hold:

$$\begin{aligned}
&2a_{n-1} - a_{n-2} + 1 \\
&= 2\frac{1}{2}(n-1)n - \frac{1}{2}(n-2)(n-1) + 1 \\
&= (n^2 - n) + \left(-\frac{1}{2}n^2 + \frac{3}{2}n - 1\right) + 1 \\
&= \frac{1}{2}n^2 + \frac{1}{2}n \\
&= \frac{1}{2}n(n+1) = a_n.
\end{aligned}$$

Therefore, if $a_n = \frac{1}{2}n(n+1)$ for all $n = 0, 1, 2, \ldots$, then $a_n = 2a_{n-1} - a_{n-2} + 1$ for all $n = 2, 3, 4, \ldots$. This is exactly what it means for the sequence defined by $a_n = \frac{1}{2}n(n+1)$ to be a solution to the recurrence relation, so $\frac{1}{2}n(n+1)$ $\boxed{\text{is a solution}}$.

5. (5 points) Find the general solution to the ordinary differential equation

$$(t^3 - t)y' \;=\; (t+2).$$

This is equivalent to finding the antiderivative of the function

$$f(t) \;=\; \frac{t+2}{t^3-t} \;=\; \frac{t+2}{t(t-1)(t+1)}$$

Applying the method of partial fractions, we seek constants a, b and c such that

$$f(t) \;=\; \frac{a}{t} + \frac{b}{t-1} + \frac{c}{t+1} \;=\; \frac{a(t-1)(t+1) + bt(t+1) + ct(t-1)}{t(t-1)(t+1)}.$$

The numerator equals $(a{+}b{+}c)t^2{+}(b{-}c)t{-}a$, and hence we have $a{+}b{+}c = 0, b{-}c = 1$, and $a = -2$. This implies $a = -2, b = 3/2, c = 1/2$, so our rational function equals

$$f(t) \;=\; \frac{1/2}{t+1} - \frac{2}{t} + \frac{3/2}{t-1}.$$

Therefore the desired antiderivative equals

$$y(t) \;=\; \int f(t)dt \;=\; \boxed{(1/2)\cdot\ln|t+1| \;-\; 2\cdot\ln|t| \;+\; (3/2)\cdot\ln|t-1| \;+\; C}.$$

Here C is an arbitrary constant.

6. (5 points) This table records the **observed frequencies** of joint outcomes for a random variable X with values x_1, x_2 and a random variable Y with values y_1, y_2.

	x_1	x_2
y_1	17	12
y_2	11	8

(a) Compute the table of **expected frequencies** under the null hypothesis H_0 that X and Y are independent. (You do not need to simply the numbers in your answer).

The row sums of the given table are 29 and 19, and the column sums are 28 and 20. The sample size is 48. Hence the table of expected frequencies is

	x_1	x_2
y_1	$\frac{29\cdot 28}{48}$	$\frac{29\cdot 20}{48}$
y_2	$\frac{19\cdot 28}{48}$	$\frac{19\cdot 20}{48}$

(b) Explain how to use the tables of observed and expected frequencies to carry out a χ^2 test. Under what circumstances will we reject the null hypothesis?

We compute the test statistic for independence on this data set as follows:

$$r = \frac{\left(17 - \frac{29\cdot 28}{48}\right)^2}{\frac{29\cdot 28}{48}} + \frac{\left(11 - \frac{19\cdot 28}{48}\right)^2}{\frac{19\cdot 28}{48}} + \frac{\left(12 - \frac{29\cdot 20}{40}\right)^2}{\frac{29\cdot 20}{48}} + \frac{\left(8 - \frac{19\cdot 20}{40}\right)^2}{\frac{19\cdot 20}{48}}.$$

For this value of r we compute the p-value to be $P\big(R \geq r \mid H_0\big)$ where P has the χ-square distribution with one degree of freedom. If this p-value is small we reject the null hypothesis of independence. If it is large, say larger than 5%, then we do not reject the null hypothesis.

7. (5 points) Determine all functions $y(t)$ that satisfy

$$y'' + 4y' + 5y = 0 \quad \text{and} \quad y(0) = 3 \text{ and } y(\pi/2) = 0$$

(Your final answer should not involve the complex number $i = \sqrt{-1}$.)

The given second-order linear ODE is # 12 on homework 9 with $a = 2$.
Its characteristic equation $\lambda^2 + 4\lambda + 5 = 0$ has two complex solutions, namely

$$\lambda = -2 \pm i.$$

This implies that the general solution to the ODE is

$$y(t) \;=\; c_1 e^{-2t}\cos(t) \;+\; c_2 e^{-2t}\sin(t),$$

where c_1 and c_2 are arbitrary constants.

By plugging in the two given values we find

$$3 \;=\; y(0) \;=\; c_1 e^0 \cos(0) \;+\; c_2 e^0 \sin(0) \;=\; c_1$$

and

$$0 \;=\; y(\pi/2) \;=\; c_1 e^{-\pi}\cos(\pi/2) \;+\; c_2 e^{-\pi}\sin(\pi/2) \;=\; c_2 e^{-\pi}.$$

Therefore $c_1 = 3$ and $c_2 = 0$.

We conclude that the problem has a unique solution, namely the function

$$\boxed{y(t) \;=\; 3e^{-2t}\cos(t)}.$$

8. (5 points) Find all solutions to the differential equation

$$y' + y^2 \cdot \cos(t) \;=\; 0.$$

This is a separable first-order ODE. We first note that the constant function $y = 0$ is a solution. For all other solutions that are define on some open subset of the real line $\mathbb{R}$, we can divide by y, and we can write

$$\int \frac{dy}{y^2} \;=\; -\int \cos(t)dt.$$

By passing to antiderivatives, we find

$$-y^{-1} \;=\; -\sin(t) + C,$$

where C is a constant. Multiplying by -1 and taking reciprocals, we find

$$y(t) \;=\; \frac{1}{\sin(t) - C}.$$

Conclusion: The set of all solutions to the given ODE consists of the zero solution $\boxed{y(t) = 0 \text{ and } y(t) \;=\; \frac{1}{\sin(t) - C}, \text{ where } C \text{ ranges over all real constants}}$.