MATH 105 - FINAL 5/16/2006 D. Geba

1. Let $(\mathbb{X}, \mathbb{S}, \nu)$ be a measure space with ν a finite measure. For A, $B \in \mathbb{S}$ we say that $A \sim B$ if and only if $\nu(A \Delta B) = 0$. Prove that:

i) $\sim$ is an equivalence relationship on $\mathbb{S}$;

ii) if $A_1 \sim A$ and $B_1 \sim B$ then $\nu(A \Delta B) = \nu(A_1 \Delta B_1)$.

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue integrable function and define

$$E_n = \{x; |f(x)| > n\}$$

Prove that $\lim_{n \to \infty} n\, \mu(E_n) = 0$.

3. Define $f : (0, \infty) \longrightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{\sqrt{x}(1 + |\log x|)}$$

Prove that $f \in L^2(0, \infty)$ but $f \notin L^p(0, \infty)$ for every $p \neq 2$, $1 \leq p < \infty$.

4. Prove that if $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are Lebesgue integrable functions, then $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$, defined by

$$h(x, y) = f(x) \cdot g(y)$$

is λ-integrable and

$$\int h \, d\lambda = \int f \, d\mu \cdot \int g \, d\mu$$

5. Let $(\phi_n)_n$ and $(\psi_n)_n$ be two orthonormal systems in $L^2[a, b]$ such that

$$\sum_{n=1}^{\infty} \|\phi_n - \psi_n\|^2 < 1$$

Prove that these systems are either both complete or incomplete.