May 17, 2004 — Math 118 — M. Rieffel

Final Examination

State your answers clearly and fully (with whole sentences, please). Include all your work.
(Total points = 100.)

1. Given operators L and H on $\ell^2(Z)$ which satisfy

$$LL^* = I, \quad HH^* = I \quad \text{and} \quad L^*L + H^*H = I,$$

15 explain carefully how the corresponding standard decomposition and reconstruction algorithms for elements of $\ell^2(Z)$ work. Show, in particular, that one obtains "perfect reconstruction".

2. a) Define what is meant by the expected value of a
6 function in $L^2(R)$, and what is meant by its dispersion about a point of R.

7 b) State the (Heisenberg) Uncertainty Principle.

3. For the Daubechies filter

$$p = (p_0, p_1, p_2, p_3) = (\sqrt{2}/8)(1+\sqrt{3},\ 3+\sqrt{3},\ 3-\sqrt{3},\ 1-\sqrt{3}) :$$

a) which are the fewest terms of $p^* * p$ that you need to
4 actually calculate in order to determine $D(p^* * p)$?

8 b) Verify that $D(p^* * p) = \delta_0$.

c) Write explicitly the scaling equation for the scaling
5 function for the above Daubechies filter.

$\phi(x) = \sum p_k \phi(2x-k)$

4. For a differentiable function, f , in $L^1(R)$ whose
11 derivative is also in $L^1(R)$, derive the formula for the Fourier transform of f' in terms of that for f . Make clear the conventions that you are using.

5. Derive carefully the algorithm (or formula) for the Fast
15 Fourier Transform on Z_6 where the inner Fourier transform is taken to be that for Z_2.

6. Let V be a finite-dimensional subspace of a vector space with inner product.

a) Explain how to construct the orthogonal projection
6 operator, P , of V onto W in terms of a given orthonormal basis for W.

b) Prove that for any v in V the point Pv is the closest
8 point in W to v.

7. a) State carefully the Shannon-Whittaker sampling
7 theorem. band limited: $f(t) = \sum_{j \in Z} f\left(\frac{2\pi j}{\Omega}\right) \frac{\sin(\Omega t - \pi j)}{\Omega t - \pi j}$

b) Give an explicit example of <u>aliasing</u>, that is, the phenomenon that if one samples less often than the
8 sampling theorem requires, then the sampling of high-frequency (parts of) a signal can appear to be the sampling of false low-frequency (parts of) a signal ("artifacts").