Math 185 (Section 3) Midterm Exam

March 4, 2003

K. Hare

NAME (printed) : ______________________
(Family Name) (First Name)

Signature : ______________________

Student Number : ______________________

(1) Do NOT open this test booklet until told to do so

(2) Do ALL your work in this test booklet

(3) SHOW ALL YOUR WORK

(4) CHECK THAT THERE ARE 6 PROBLEMS

(5) NO CALCULATORS

(6) No pushing, biting, or hitting

1	2	3	4	5	6	TOTAL

1 a: (3 pts) Define what it means for a set D to be i) open, ii) closed, iii) a domain

i) open: A set D is open if all points $z \in D$ are interior points.

ii) closed: A set D is closed if D contains its boundary.

iii) a domain: A set D is a domain if it is an open connected set.

b: (4 pts) Find the principle root of

$$\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^{\frac{1}{3}}$$

$$\begin{aligned}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^{\frac{1}{3}} &= e^{\frac{1}{3}\mathrm{Log}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)} \\ &= e^{\frac{1}{3}(\ln(1)+\frac{3\pi}{4}i)} \\ &= e^{\frac{1}{3}(\frac{3\pi}{4}i)} \\ &= e^{\frac{\pi}{4}i} \\ &= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\end{aligned}$$

c: (3 pts) Find in Cartesian (rectangular) co-ordinates:

$$(-1+\sqrt{3}i)^{100}$$

$$\begin{aligned}
(-1+\sqrt{3}i)^{100} &= \left(2e^{\frac{2\pi i}{3}}\right)^{100} \\
&= 2^{100}e^{\frac{200\pi i}{3}} \\
&= 2^{100}e^{\frac{2\pi i}{3}+66\pi i} \\
&= 2^{100}e^{\frac{2\pi i}{3}} \\
&= 2^{99}(-1+\sqrt{3}i)
\end{aligned}$$

2 a: (3 pts) Find a harmonic conjugate for $u(x,y) = x + 2xy$.

Notice that $u_x(x,y) = 2y+1 = v_y(x,y)$ *which implies that* $v(x,y) = y^2 + y + \phi(x)$.

Notice that $u_y(x,y) = 2x = -v_x(x,y) = -\phi'(x)$ *which implies that* $\phi(x) = -x^2 + c$

Thus we have that

$$v(x,y) = y^2 - x^2 + y + c$$

is the harmonic conjugate of $u(x,y)$.

b: (3 pts) Find the principle value of i^i.

$$\begin{aligned} i^i &= e^{i\mathrm{Log}(i)} \\ &= e^{i(\log(1)+\frac{i\pi}{2})} \\ &= e^{i\frac{i\pi}{2}} \\ &= e^{\frac{-\pi}{2}} \end{aligned}$$

c: (4 pts) Find the following limits, or state why they do not exist

i) $\lim_{z\to\infty} \frac{z^2+1}{1-iz^2}$,

$$\lim_{z\to\infty} \frac{z^2+1}{1-iz^2} = \lim_{z\to\infty} \frac{1+1/z^2}{1/z^2-i} = \frac{1}{-i} = i$$

ii) $\lim_{z\to\infty} \sin(z)$,

If the limit existed, it would exist along any line going towards infinity. We know that along the x-axis that $\sin(z)$ *oscillates between* -1 *and* 1*. Thus the limit does not exist along the real axis. Thus the limit does not exist.*

iii) $\lim_{z\to\infty} \mathrm{Log}(z)$,

$$\lim_{z\to\infty} \mathrm{Log}(z) = \lim_{z\to\infty} \log(|z|) + \arg z i = \infty$$

iv) $\lim_{z\to\infty} \frac{1}{z^2+1}$

$$\lim_{z\to\infty} \frac{1}{z^2+1} = \lim_{z\to\infty} \frac{1/z^2}{1+1/z^2} = 0/1 = 0$$

3 a: (3 pts) Let $f(z) = \bar{z}$. Use the Cauchy-Riemann equations to show that $f'(z)$ does not exists for all complex numbers z.

Notice that $f(z) = x - yi$. *Thus* $u_x = 1 \neq -1 = v_y$. *Thus the Cauchy-Riemann equations do not hold. Thus* $f'(z)$ *does not exist anywhere.*

b: (5 pts) Let $f(z) = \bar{z}$. Use the formal definition of the derivative to show that $f'(z)$ does not exists for all complex numbers z.

The formal definition of the limit is

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z} + \bar{\Delta} z - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{\Delta} z}{\Delta z}$$

We see that if we approach along the real axis, this limit is 1. *If we approach along the imaginary axis, this limit is* -1. *Thus this limit does not exist. Thus* $f'(z)$ *does not exists anywhere.*

c: (4 pts) Using the formal definition of a limit, show that

$$\lim_{z \to i} 1 + 2\bar{z} = 1 - 2i.$$

Pick $\delta = \frac{\epsilon}{2}$. *Then we get that*

$$\begin{aligned} |z - i| &< \delta = \tfrac{\epsilon}{2} \\ \Rightarrow \quad |2z - 2i| &< \epsilon \\ \Rightarrow \quad |2\bar{z} + 2i| &< \epsilon \\ \Rightarrow \quad |2\bar{z} + 1 - (1 - 2i)| &< \epsilon \end{aligned}$$

Which gives the desired result.

4 a: (4 pts) Let $v(x,y)$ be a harmonic conjugate of $u(x,y)$. Why must $U(x,y) = e^{u(x,y)} \cos(v(x,y))$ be harmonic?

If $v(x,y)$ is the harmonic conjugate of $u(x,y)$, then we know that $u(x,y) + iv(x,y)$ is analytic. Thus we know that $e^{u(x,y)+iv(x,y)}$ is analytic. Notice that

$$e^{u(x,y)+iv(x,y)} = e^{u(x,y)} \cos(v(x,y)) + ie^{u(x,y)} \sin(v(x,y)).$$

But the real part of analytic functions are harmonic. Thus

$$e^{u(x,y)} \cos(v(x,y))$$

is harmonic.

b: (4 pts) Let f be an entire function such that $f(\bar{z}) = -\overline{f(z)}$. Show that f must be purely imaginary on the real axis.

Soln 1: Let $F(z) = if(z)$. Thus we see that

$$F(\bar{z}) = if(\bar{z}) = -i\overline{f(z)} = \overline{if(z)} = \overline{F(z)}$$

Thus $F(z)$ satisfies the reflection princple, and is real on the real line. Thus $f(z)$ is purely imaginary on the real line.

Soln 2: Notice that on the real line that $\bar{z} = z$. This we have

$$-\overline{f(z)} = f(\bar{z}) = f(z)$$

This tells us that $f(z)$ is purely imaginary.

c: (4 pts) Show that for all z_0, z_1 in the complex numbers that $\int_{z_0}^{z_1} f(z)\ dz$ is path independent if and only if

$$\int_C f(z)\ dz = 0$$

for all closed contours C in the complex plane.

$\Rightarrow$ *Assume that* $\int_{z_0}^{z_1} f(z)\ dz$ *is path independant. Let* C *be a closed contour. Let* $z_0 = z_1$ *be a point on the contour. Then*

$$\int_C f(z)\ dz = \int_{z_0}^{z_1} f(z)\ dz = \int_{z_0}^{z_0} f(z)\ dz = 0$$

and we are done.

$\Leftarrow$ *Assume that* $\int_C f(z)\ dz = 0$ *for all closed contours. Consider two paths* C_1 *and* C_2 *between* z_0 *and* z_1. *Let* C *be the close contour created by matching up* C_1 *to* $-C_2$. *Thus we have that:*

$$0 = \int_C f(z)\ dz = \int_{C_1} f(z)\ dz - \int_{C_2} f(z)\ dz$$

and this implies that

$$\int_{C_1} f(z)\ dz = \int_{C_2} f(z)\ dz$$

and hence integration is path independant.

5 a: (5 pts) Show that

$$\sinh^{-1}(z) = \log(z + \sqrt{z^2+1})$$

Let $w = \sinh^{-1}(z)$. *Then we know that*

$$\begin{aligned} \sinh(w) &= z \\ \Rightarrow \quad \tfrac{e^w - e^{-w}}{2} &= z \\ \Rightarrow \quad e^w - e^{-w} &= 2z \\ \Rightarrow \quad e^w - 2z - e^{-w} &= 0 \\ \Rightarrow \quad e^{2w} - 2ze^w - 1 &= 0 \end{aligned}$$

Thus by the quadratic formula we get

$$\begin{aligned} e^w &= \tfrac{2z \pm \sqrt{4z^2+4}}{2} \\ \Rightarrow \quad e^w &= z \pm \sqrt{z^2+1} \\ \Rightarrow \quad w &= \log(z \pm \sqrt{z^2+1}) \end{aligned}$$

Which is the desired result.

b: (3 pts) Using the equation from part **a** find $\sinh^{-1}(i)$.

$$\sinh^{-1}(i) = \log(i \pm \sqrt{-1+1}) = \log(i) = \frac{\pi}{2}i + 2\pi i k$$

where k *is any integer*

6 a: (4 pts) Let

$$C = \begin{cases} t + ti & \text{when } 0 \le t \le 1 \\ t + 2i - ti & \text{when } 1 \le t \le 2 \\ 4 - t & \text{when } 2 \le t \le 4 \end{cases}$$

Find

$$\int_C \cos(z)\ dz$$

Notice that C is a closed contour. Notice that $\cos(z)$ has an continuous anti-derivative in the complex plane. Thus

$$\int_C \cos(z)\ dz = 0$$

b: (4 pts) Using $C = \{e^{it} :\ 0 \le t \le 4\pi\}$ find

$$\int_C \frac{1}{z}\ dz$$

$$\int_C \frac{1}{z}\ dz = \int_0^{4\pi} \frac{1}{e^{it}} ie^{it}\ dt = \int_0^{4\pi} i\ dt = 4\pi i$$