

Department of Mathematics, University of California,
Berkeley

Math 214

Alan Weinstein, Spring 2001

Take Home Final Examination

Due in class (61 Evans Hall) at 11:15 AM, Tuesday, May 8, 2001

Instructions. You may use your class notes and the Spivak text, but no other references. You should consult nobody except A.W. about the exam. Please send questions about the exam to alanw@math.berkeley.edu and not to the course emailing list. If I learn of errors or imprecisions on the exam, I will send my corrections to the emailing list (and they will then appear at <http://socrates.berkeley.edu/~alanw/mail-archive.214> as well).

Do all of the problems. If you have trouble with one part of a problem, you may still use its result to try the following parts. Unless otherwise specified, all manifolds, maps, flows, actions, . . . are C^∞ .

1. Let X be a complete vector field on a manifold M , T a positive real number. An integral curve $c : \mathbb{R} \rightarrow M$ of X is said to be **periodic with least period T** if $c(T) = c(0)$ but $c(t) \neq c(0)$ for $t \in (0, T)$.

(A) Prove that an integral curve with least period T is injective on $[0, T)$.

(B) Prove that, if every integral curve of X is periodic with the same least period T , then the flow of X determines a free action of the circle group $\mathbb{R}/T\mathbb{Z}$ on M .

(C) Given a free action of the circle group $S^1 = \mathbb{R}/T\mathbb{Z}$ on M , prove that the set M/S^1 of orbits carries a unique differentiable structure such that the projection π from M to the orbit space is a submersion.

(D) For a free action of S^1 generated by the vector field X , a **connection** is a 1-form ϕ on M which is invariant under the action and which has the property that $\phi(X)$ is identically equal to 1. Prove that, if ϕ is a connection, then the 2-form $d\phi$ is the pullback π^* of a well defined *closed* form ω on

the quotient manifold M/S^1 . (The form ω is called the **curvature** of the connection.)

(E) A connection is called **flat** if and only if its curvature is identically zero. Prove that ϕ is flat if and only if the distribution $\ker\phi$ is integrable.

(F) Prove that the cohomology class of the curvature is independent of the chosen connection. Deduce from this that, if M is compact and 3-dimensional and carries a free action of the circle group with a connection whose curvature is nowhere zero, then there is no flat connection for this action.

(G) The integral curves of a nonzero left-invariant vector field on the group $SO(3)$ all have the same least period. Why? What is the space of orbits? For such a vector field, find a left-invariant connection and compute its curvature. Is it nowhere zero? Everywhere zero?

(H) Is there a vector field on the Klein bottle which generates a free circle action?

2. Using the fact that de Rham cohomology can be computed with the subcomplex of forms invariant under the action of a compact, connected group, compute the de Rham cohomology of S^2 by using forms invariant under the action of $SO(3)$.

3. Again using invariant forms, compute the de Rham cohomology of the torus T^n for any n .

4. Let M be an orientable manifold. Show that, if $I : C_c^\infty(M) \rightarrow \mathbb{R}$ is a linear map such that $I(f \circ \phi) = I(f)$ for every $f \in C_c^\infty(M)$ and every orientation preserving diffeomorphism $\phi : M \rightarrow M$, then I must be the zero map. (Actually, a very weak hypothesis on M is necessary for this statement to be true. What is it?)