Prof. Bjorn Poonen
October 16, 2001

MATH 55 MIDTERM (white)

Do not write your answers on this sheet. Instead please write your name, your student ID, your TA's name, your section time, "white," and all your answers in your blue books. Total: 100 pts., 75 minutes.

(1) (5 pts. each) For each of (a)-(g) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.) No explanations are required in this problem.

(a) The value of $(-141) \bmod 8$ is -5.

(b) The function $3n^2 \log n + 5n(\log n)^4$ is $O(n^3)$.

(c) The ceiling function $f(x) = \lceil x \rceil$, considered as a function from $\mathbb{R}$ to $\mathbb{R}$, has an inverse function.

(d) The set $\{1,2,3\} \times \mathbb{Z}$ is countable.

(e) The proposition $\forall x \exists y (x \leq y)$ is true, when the universe of discourse is the set of natural numbers.

(f) The numbers $35, 36, 37$ are pairwise relatively prime.

(g) There are infinitely many integers x satisfying both $x \equiv 22 \pmod{99}$ and $x \equiv 25 \pmod{100}$. (Hint: you don't need to solve this system.)

(2) (7 pts.) Find the hexadecimal (base 16) expansion of $(270)_{10}$. (Show your work.)

(3) (15 pts.) Find *two* integer solutions to the congruence $31x \equiv 2 \pmod{100}$. (Explain how you are solving the problem as you go along.)

(4) (10 pts.) Find the prime factorization of 565. (Explain how you are solving this problem. If you claim that a number is prime, explain why.)

(5) (8 pts.) Is the compound proposition $(p \to q) \land (q \to r)$ logically equivalent to $p \to r$? Explain.

(6) (15 pts.) Suppose that f is an injective function from the set A to the set B, and that S is a subset of A. Prove that $f^{-1}(f(S)) = S$. (Recall that "injective" means the same thing as "one-to-one", and recall that for $T \subseteq B$, $f^{-1}(T)$ is defined as $\{x \in A : f(x) \in T\}$.)

(7) (10 pts.) The sequence $a_0, a_1, a_2, \ldots$ is defined recursively by $a_0 = 1$ and $a_{n+1} = 3a_n - 1$ for $n \geq 0$. Prove that $a_n = (3^n + 1)/2$ for all integers $n \geq 0$.

This is the end! At this point, you may want to look over this sheet to make sure you have not omitted any problems. Check that your answers make sense! Please take this sheet with you as you leave.

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MATH 55 MIDTERM (yellow)

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(1) (5 pts. each) For each of (a)-(g) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.) No explanations are required in this problem.

(a) The set $\{0, 1\}^*$ of bit strings of finite length is a countable set.

(b) The proposition $\exists x \forall y (x \geq y)$ is true, when the universe of discourse is the set of natural numbers.

(c) The numbers $34, 35, 36$ are pairwise relatively prime.

(d) There are infinitely many integers x satisfying both $x \equiv 12 \pmod{99}$ and $x \equiv 16 \pmod{100}$. (Hint: you don't need to solve this system.)

(e) The function $3n^2 \log n + 5n(\log n)^4$ is $O(n^3)$.

(f) The floor function $f(x) = \lfloor x \rfloor$, considered as a function from $\mathbb{R}$ to $\mathbb{R}$, has an inverse function.

(g) The value of $(-133) \bmod 9$ is -7.

(2) (8 pts.) Is the compound proposition $(p \to q) \land (q \to r)$ logically equivalent to $p \to r$? Explain.

(3) (10 pts.) Find the prime factorization of 539. (Explain how you are solving this problem. If you claim that a number is prime, explain why.)

(4) (7 pts.) Find the hexadecimal (base 16) expansion of $(269)_{10}$. (Show your work.)

(5) (15 pts.) Find *two* integer solutions to the congruence $31x \equiv 4 \pmod{100}$. (Explain how you are solving the problem as you go along.)

(6) (10 pts.) The sequence $a_0, a_1, a_2, \ldots$ is defined recursively by $a_0 = 1$ and $a_{n+1} = 3a_n - 1$ for $n \geq 0$. Prove that $a_n = (3^n + 1)/2$ for all integers $n \geq 0$.

(7) (15 pts.) Suppose that f is a surjective function from the set A to the set B, and that S is a subset of B. Prove that $f(f^{-1}(S)) = S$. (Recall that "surjective" means the same thing as "onto", and recall that $f^{-1}(S)$ is defined as $\{x \in A : f(x) \in S\}$.)

This is the end! At this point, you may want to look over this sheet to make sure you have not omitted any problems. Check that your answers make sense! Please take this sheet with you as you leave.

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