Prof. Bjorn Poonen
December 14, 2001

MATH 55 FINAL (white)

Do not write your answers on this sheet. Instead please write your name, your student ID, your TA's name, your section time, "white," and all your answers in your blue books. IMPORTANT: Write your answers to problems 1–12 on the first page or two of your blue book, without the calculations you did to get the answers. To guarantee full credit and to qualify for partial credit, you should show your work on *later* pages of the blue book, labelled by problem number. Total: 200 pts., 170 minutes.

(1) (5 pts. each) For each of (a)-(e) below: If the statement is true (always), write TRUE. Otherwise write FALSE. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.) No explanations are required in this problem.

(a) If a function $f(n)$ is $\Theta(2^n)$ as $n \to \infty$, then $f(n)$ is also $o(n!)$ as $n \to \infty$.

(b) For any predicate $P(x)$, the propositions $\neg\forall x P(x)$ and $\exists x P(x)$ are logically equivalent.

(c) Exactly half of the 4-element subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ contain the number 5.

(d) If A, B, and C are pairwise disjoint sets (possibly infinite), and $|A \cup B| = |A \cup C|$, then $|B| = |C|$.

(e) The statement $p\{S\}q$ is true, where p is the assertion "$n = -1$", q is the assertion "$k = -1$", and S is the program segment consisting of the following six lines:

```
k := 1
while n ≠ 0
begin
        n := n − 1
        k := n * k
end
```

(2) (15 pts.) For each integer $n \geq 1$, let a_n denote the number of strings of n letters in which every "q" is immediately followed by a "u". Find a recurrence relation and initial condition(s) that determine the sequence $a_1, a_2, \ldots$. (Do not try to find an explicit formula for a_n.)

(3) (20 pts.) A sequence $a_0, a_1, \ldots$ satisfies $a_0 = 5$, $a_1 = 14$, and

$$a_n = a_{n-1} + 6a_{n-2} + 5(3^n)$$

for $n \geq 2$. Find an explicit formula for a_n.

(4) (15 pts.) A fair coin is flipped. If it comes up heads, then two more fair coins are flipped; if instead the first coin comes up tails, then only one more fair coin is flipped. (Thus the total number of coin flips is either 3 or 2.) What is the probability that there is exactly one tail in all the flips?

(5) (10 pts. each) A game is played by rolling a fair 6-sided die 21 times. The player scores two points for each 6 that is rolled and one point for each 5 that is rolled.

(a) What is the expected value of the total score?
(b) What is the standard deviation of the total score?

(6) (10 pts.) When the 4-combinations of $\{1,2,3,4,5,6\}$ are listed in lexicographic order, which 4-combination is eighth on the list?

(7) (10 pts.) How many cards must be taken from a standard deck to guarantee that there will exist some suit such that the selected cards include at least four cards of that suit? (In a standard deck, there are four suits, and 13 cards of each suit, for a total of 52 cards.)

(8) (15 pts.) Alice stands at the bottom of a staircase. How many ways can she take a sequence of 5 steps, where each step must be either a step up or a step down? (Be careful: if at any time she is at the bottom of the staircase, her next step must be upwards, obviously! Assume that the staircase has more than 5 steps.)

(9) (10 pts.) For each nonnegative integer n, let a_n be the number of solutions (x, y, z) to the equation $x + 3y + 5z = n$ in nonnegative integers. Express the generating function of the sequence $a_0, a_1, a_2, \ldots$ in closed form.

(10) (20 pts.) How many functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5,6,7\}$ have the property that $|f^{-1}(\{1,2\})| = 2$? (Note: $f^{-1}(\{1,2\})$ denotes the inverse image of the set $\{1,2\}$ under f.)

(11) (15 pts.) How many positive integers less than or equal to 150 are relatively prime to 30?

(12) (10 pts.) How many times will the command

$$\mathit{wastetime}(6)$$

print "Hi" if the procedure *wastetime* is defined by the pseudocode below?

```
procedure wastetime(n: positive integer)
if n = 1 then
      print "Hi"
else for i := 1 to n − 1
      wastetime(i)
```

(13) (15 pts.) Prove the identity

$$\binom{0}{6} + \binom{1}{6} + \binom{2}{6} + \cdots + \binom{n-1}{6} = \binom{n}{7}$$

for all integers $n \geq 1$. (Suggestion: use induction.)

This is the end! At this point, you may want to look over the exam to make sure you have not omitted any problems. Check that your answers make sense! Please take this exam with you as you leave.

Prof. Bjorn Poonen
December 14, 2001

MATH 55 FINAL (yellow)

Do not write your answers on this sheet. Instead please write your name, your student ID, your TA's name, your section time, "yellow," and all your answers in your blue books. IMPORTANT: Write your answers to problems 1–12 on the first page or two of your blue book, without the calculations you did to get the answers. To guarantee full credit and to qualify for partial credit, you should show your work on *later* pages of the blue book, labelled by problem number. Total: 200 pts., 170 minutes.

(1) (5 pts. each) For each of (a)-(e) below: If the statement is true (always), write TRUE. Otherwise write FALSE. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistiguishable.) No explanations are required in this problem.

(a) Exactly half of the 3-element subsets of $\{1,2,3,4,5,6,7\}$ contain the number 5.

(b) If A, B, and C are pairwise disjoint sets (possibly infinite), and $|A \cup B| = |A \cup C|$, then $|B| = |C|$.

(c) If a function $f(n)$ is $O(2^n)$ as $n \to \infty$, then $f(n)$ is also $o(n!)$ as $n \to \infty$.

(d) For any predicate $P(x)$, the propositions $\neg\exists x P(x)$ and $\forall x P(x)$ are logically equivalent.

(e) The statement $p\{S\}q$ is true, where p is the assertion "$n = -1$", q is the assertion "$k = -1$", and S is the program segment consisting of the following six lines:

```
k := 1
while n ≠ 0
begin
        n := n − 1
        k := n * k
end
```

(2) (10 pts.) When the 4-combinations of $\{1,2,3,4,5,6\}$ are listed in lexicographic order, which 4-combination is seventh on the list?

(3) (10 pts.) How many cards must be taken from a standard deck to guarantee that there will exist some suit such that the selected cards include at least three cards of that suit? (In a standard deck, there are four suits, and 13 cards of each suit, for a total of 52 cards.)

(4) (15 pts.) Alice stands at the bottom of a staircase. How many ways can she take a sequence of 5 steps, where each step must be either a step up or a step down? (Be careful: if at any time she is at the bottom of the staircase, her next step must be upwards, obviously! Assume that the staircase has more than 5 steps.)

2

(5) (10 pts.) For each nonnegative integer n, let a_n be the number of solutions (x, y, z) to the equation $x + 2y + 4z = n$ in nonnegative integers. Express the generating function of the sequence $a_0, a_1, a_2, \ldots$ in closed form.

(6) (20 pts.) How many functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5,6,7\}$ have the property that $|f^{-1}(\{1,2\})| = 3$? (Note: $f^{-1}(\{1,2\})$ denotes the inverse image of the set $\{1,2\}$ under f.)

(7) (15 pts.) How many positive integers less than or equal to 300 are relatively prime to 30?

(8) (10 pts.) How many times will the command

$$wastetime(7)$$

print "Hi" if the procedure *wastetime* is defined by the pseudocode below?

```
procedure wastetime(n: positive integer)
if n = 1 then
      print "Hi"
else for i := 1 to n − 1
      wastetime(i)
```

(9) (15 pts.) For each integer $n \geq 1$, let a_n denote the number of strings of n letters in which every "q" is immediately followed by a "u". Find a recurrence relation and initial condition(s) that determine the sequence $a_1, a_2, \ldots$. (Do not try to find an explicit formula for a_n.)

(10) (20 pts.) A sequence $a_0, a_1, \ldots$ satisfies $a_0 = 4$, $a_1 = 16$, and

$$a_n = a_{n-1} + 6a_{n-2} + 5(3^n)$$

for $n \geq 2$. Find an explicit formula for a_n.

(11) (15 pts.) A fair coin is flipped. If it comes up heads, then two more fair coins are flipped; if instead the first coin comes up tails, then only one more fair coin is flipped. (Thus the total number of coin flips is either 3 or 2.) What is the probability that there is exactly one head in all the flips?

(12) (10 pts. each) A game is played by rolling a fair 6-sided die 21 times. The player scores two points for each 6 that is rolled and one point for each 5 that is rolled.

(a) What is the expected value of the total score?

(b) What is the standard deviation of the total score?

(13) (15 pts.) Prove the identity

$$\binom{0}{7} + \binom{1}{7} + \binom{2}{7} + \cdots + \binom{n-1}{7} = \binom{n}{8}$$

for all integers $n \geq 1$. (Suggestion: use induction.)

This is the end! At this point, you may want to look over the exam to make sure you have not omitted any problems. Check that your answers make sense! Please take this exam with you as you leave.