Department of Mathematics, University of California, Berkeley

Math 214

Alan Weinstein, Fall 2001

Take Home Final Examination

Due in class (2 Evans Hall) at 11:15 AM, Thursday, 12/6/01 (GROUP 1), or in 825 Evans Hall (under the door) by 11:15 AM, Tuesday, 12/11/01 (GROUP 2).

Instructions. You may use your class notes and the Spivak text, but no other references. You should consult nobody except A.W. about the exam. Please send questions about the exam to `alanw@math.berkeley.edu` and not to the course emailing list. If I learn of errors or imprecisions on the exam, I will send my corrections to the emailing list after the exam is distributed to Group 2 (and they will then appear at `http://socrates.berkeley.edu/~alanw/mail-archive.214` as well). Before then, I will send corrections to members of Group 1 who send me an email request.

Do all of the problems. If you have trouble with one part of a problem, you may still use its result to try the following parts. Unless otherwise specified, all manifolds, maps, flows, actions, ... are C^∞.

1. Let M be the quotient of $\mathbb{R}^n \setminus \{0\}$ by the cyclic group of transformations generated by the map $x \mapsto 7x$. Prove that M (with the quotient topology) is a manifold by exhibiting a set of charts and showing that the overlap maps are smooth. For simplicity, you may use charts whose images are arbitrary open subsets of euclidean space. Then show that M is diffeomorphic to a product of two manifolds, neither of which is a single point.

2. Find an automorphism of the exterior algebra $\wedge\mathbb{R}^{3*}$ which does *not* leave the subspace $\wedge^1\mathbb{R}^{3*} = \mathbb{R}^{3*}$ invariant. [It may (or may not!) help you to think of this automorphism as corresponding to a diffeomorphism of the "purely odd" supermanifold $\Pi\mathbb{R}^3$, of which $\wedge\mathbb{R}^{3*}$ is the "algebra of smooth functions."]

3. Let $R : G \times M \to M$ be a smooth action of the Lie group G on the manifold M. For each element v of the Lie algebra $\mathfrak{g}$ of G, define the vector field $r(v)$ by $r(v)(p) = c'(0)$, where c is the curve $t \mapsto R(\exp tv, p)$. Prove that $r(v)$ is a smooth vector field on M, and that r is a Lie algebra antihomomorphism from $\mathfrak{g}$ to the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on M. (Don't forget to show that r is linear.) We call r the **infinitesimal action** associated to the action R. Prove that, if two actions of a connected Lie group G on M give rise to the same infinitesimal action, then they are equal.

4. Show that the 3-dimensional Lie algebra of vector fields on $\mathbb{R}$ generated by $\partial/\partial x$, $x\partial/\partial x$, and $x^2\partial/\partial x$ is *not* the image of the infinitesimal action associated to a group action on $\mathbb{R}$.

5. Find a 3-dimensional Lie subalgebra of the Lie algebra of vector fields on the circle S^1.

6. A 2-form ω on a manifold M is called **nondegenerate** if the map $\tilde{\omega} : TM \to T^*M$ defined by $\tilde{\omega}(v) = v \lrcorner\, \omega$ is an isomorphism. Prove that ω is nondegenerate if and only if the dimension of M is even and the "wedge power" $\omega^{\dim M/2}$ is nowhere zero.

7. A **symplectic structure** on a manifold M is a nondegenerate closed 2-form. Prove, using facts from the book about integration and de Rham cohomology, that the only spheres S^k which admit symplectic structures are S^0 and S^2. (It is known, but much harder to prove, that S^6 is the only other sphere which admits a nondegenerate 2-form.)