Birational classification of algebraic varieties

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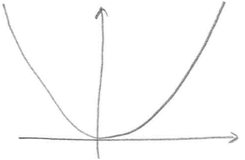
- Algebraic Geometry is the study of geometric objects defined by polynomial equations.
- In this talk we will consider complex varieties.
- For example an **affine variety** $X = V(p_1, \ldots, p_r) \subset \mathbb{C}^N$ is defined as the vanishing set of polynomial equations $p_1, \ldots, p_r \in \mathbb{C}[x_1, \ldots, x_N]$.
- A familiar example is $\{y - x^2 = 0\} \subset \mathbb{C}^2$ which corresponds to a sphere minus a point.
- It is often convenient to consider compact varieties in projective space.

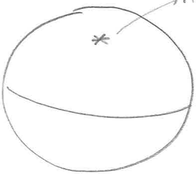
Projective space

- **Projective space** $\mathbb{P}^N_{\mathbb{C}} \supset \mathbb{C}^N$ is a natural compactification obtained by adding the hyperplane at infinity $H = \mathbb{P}^N_{\mathbb{C}} \setminus \mathbb{C}^N \cong \mathbb{P}^{N-1}_{\mathbb{C}}$.
- It is defined by $\mathbb{P}^N_{\mathbb{C}} = (\mathbb{C}^{N+1} \setminus \bar{0})/\mathbb{C}^*$ so that $(c_0, \ldots, c_N) \sim (\lambda c_0, \ldots, \lambda c_N)$ for any non-zero constant $\lambda \in \mathbb{C}^*$. The equivalence class of $(c_0, \ldots, c_N)$ is denoted by $[c_0 : \ldots : c_N]$.
- $\mathbb{C}^N$ corresponds to $\{[1 : c_1 : \ldots : c_N] | c_i \in \mathbb{C}\}$ and H to $\mathbb{P}^{N-1} \equiv \{[0 : c_1 : \ldots : c_N] | c_i \in \mathbb{C}\}/\sim$.
- We then consider projective varieties $X \subset \mathbb{P}^n$ defined by homogeneous polynomials $P_1, \ldots, P_r \in \mathbb{C}[x_0, \ldots, x_N]$.
- Note that is P is homogeneous, then $P(\lambda c_0, \ldots, \lambda c_N) = 0$ iff $P(c_0, \ldots, c_N) = 0$.
- For example:

$Y = X^2$ Affine variety in $\mathbb{C}^2$



$\mathbb{R}^2$

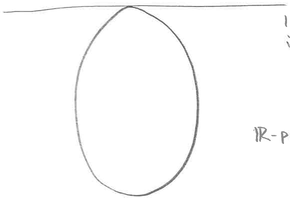


Missing pt at infinity

$\mathbb{C}^2$

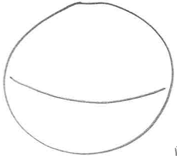
$YZ = X^2$ subvariety of $\mathbb{P}^2_{\mathbb{C}}$

$[0;1:0]$



line at infinity

$\mathbb{R}$-picture



Riemann Sphere

$\mathbb{P}^2_{\mathbb{C}}$

- From now on we consider **projective varieties**

$$X = V(P_1, \ldots, P_r) \subset \mathbb{P}^N_{\mathbb{C}}$$

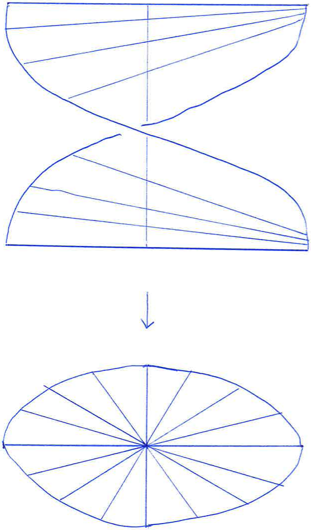
where $P_i \in \mathbb{C}[x_0, \ldots, x_n]$ are homogeneous polynomial equations and $\mathbb{P}^N_{\mathbb{C}} = (\mathbb{C}^{N+1} \setminus \bar{0})/\mathbb{C}^*$ is N-dimensional **projective space**.
- For any affine variety $X = V(p_1, \ldots, p_r) \subset \mathbb{C}^N$ we obtain a projective variety $\bar{X} = V(P_1, \ldots, P_r) \subset \mathbb{P}^N_{\mathbb{C}}$ where $P_i = p_i(x_1/x_0, \ldots, x_N/x_0)x_0^{\deg p_i}$.
- Typically we will assume that $X \subset \mathbb{P}^N_{\mathbb{C}}$ is irreducible and smooth, hence a complex manifold of dimension $d = \dim X$.
- The closed subsets in the **Zariski topology** are zeroes of polynomial equations.
- $\mathbb{C}(X) = \{P/Q \ s.t. \ Q|_X \not\equiv 0\}$ is the **field of rational functions**.

Birational equivalence

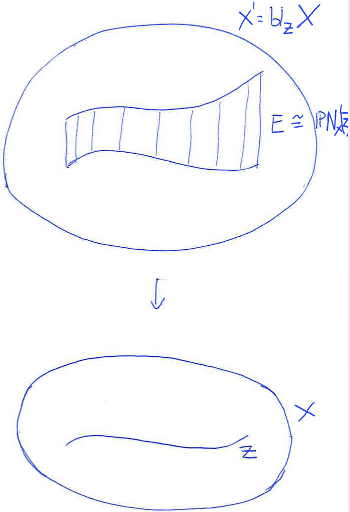
- Two varieties are **birational** if they have isomorphic open subsets.
- It is easy to see that two varieties are birational if they have the same field of rational functions $\mathbb{C}(X) \cong \mathbb{C}(Y)$.
- Recall that by **Hironaka's theorem on the resolution of singularities** (1964), every variety $X \subset \mathbb{P}^N_{\mathbb{C}}$ is birational to a smooth variety.
- More precisely there is a finite sequence of blow ups along smooth subvarieties

$$X' = X_n \to X_{n-1} \to \ldots X_1 \to X$$

  such that X' is smooth.
- If $Z \subset X$ are smooth varieties, then the **blow up** $\mathrm{bl}_Z(X) \to X$ of X along Z replaces the subset $Z \subset X$ by the codimension 1 subvariety $E = \mathbb{P}(N_Z X)$.
- We say that E is an **exceptional divisor**.



Blow up a point in $\mathbb{C}^2$



Blow up a curve in a threefold

Canonical ring

- The geometry of varieties $X \subset \mathbb{P}^N_{\mathbb{C}}$ is typically studied in terms of the **canonical line bundle** $\omega_X = \wedge^{\dim X} T_X^\vee$.
- A section $s \in H^0(\omega_X^{\otimes n})$ can be written in local coordinates as

$$f(z_1, \ldots, z_n)(dz_1 \wedge \ldots \wedge dz_n)^{\otimes m}.$$

- Of particular importance is the **canonical ring**

$$R(\omega_X) = \bigoplus_{n \geq 0} H^0(\omega_X^{\otimes n})$$

  a birational invariant of smooth projective varieties.
- The **Kodaira dimension** of X is defined by

$$\kappa(X) := \mathrm{tr.deg.}_{\mathbb{C}} R(\omega_X) - 1 \in \{-1, 0, 1, \ldots, d = \dim X\}.$$

- Note that complex projective manifolds have no non-constant global holomorphic functions, so it is natural to consider global sections of line bundles.
- There is only one natural choice: the canonical line bundle!

Canonical ring of hypersurfaces

- For example, if $X = \mathbb{P}^N_{\mathbb{C}}$ then $\omega_X = \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(-N-1)$.
- If $X_k \subset \mathbb{P}^N_{\mathbb{C}}$ is a smooth hypersurface of degree k, then $\omega_{X_k} = \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(k-N-1)|_{X_k}$.
- Here $\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(l)$ is the line bundle corresponding to homogeneous polynomials of degree l and $\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(l)|_{X_k}$ is the line bundle obtained by restriction to $X_k \subset \mathbb{P}^N_{\mathbb{C}}$.
- It is easy to see that if $k \leq N$ then $R(\omega_{X_k}) \cong \mathbb{C}$ and so $\kappa(X_k) = -1$,
- if $k = N+1$, then $R(\omega_{X_k}) \cong \mathbb{C}[t]$ and so $\kappa(X_k) = 0$, and
- if $k \geq N+2$, then $\kappa(X_k) = \dim X_k$.
- Eg. if $k = N+2$, then $\mathbb{C}[x_0, \ldots, x_n] \twoheadrightarrow R(\omega_{X_k})$.

Canonical ring of curves

- When $d = \dim X = 1$ we say that X is a curve and we have 3 cases:
- $\kappa(X) = -1$: Then $X \cong \mathbb{P}^1_{\mathbb{C}}$ is a **rational curve**. Note that $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and so $R(\omega_X) \cong \mathbb{C}$.
- $\kappa(X) = 0$: Then $\omega_X \cong \mathcal{O}_X$ and X is an **elliptic curve**. There is a one parameter family of these given by the equations

$$x^2 = y(y-1)(y-s).$$

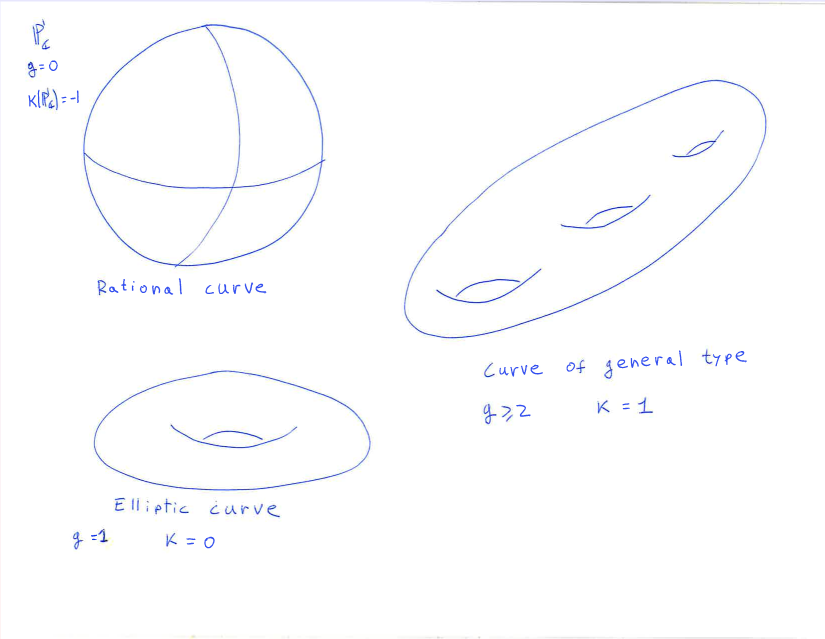
- In this case $H^0(\omega_X^{\otimes m}) \cong H^0(\mathcal{O}_X) \cong \mathbb{C}$ and so $R(\omega_X) \cong \mathbb{C}[t]$.

Canonical ring of curves

- If $\kappa(X) = 1$, then we say that X is a curve of **general type**. These are Riemann surfaces of genus $g \geq 2$.
- For any $g \geq 2$ they belong to a $3g - 3$ dimensional irreducible algebraic family. We have $\deg(\omega_X) = 2g - 2 > 0$.
- By Riemann Roch, it is easy to see that $\omega_X^{\otimes m}$ is **very ample** for $m \geq 3$. This means that if $s_0, \ldots, s_N$ are a basis of $H^0(\omega_X^{\otimes m})$, then

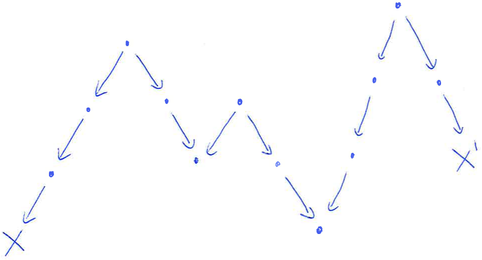
$$\phi_m : X \to \mathbb{P}^N, \qquad x \to [s_0(x) : s_1(x) : \ldots : s_N(x)]$$

  is an embedding.
- Thus $\omega_X^{\otimes m} \cong \phi_m^* \mathcal{O}_{\mathbb{P}^N}(1)$ and in particular $R(\omega_X)$ is finitely generated.
- In fact $X \cong \mathrm{Proj} R(\omega_X)$ is the variety defined by the generators and relations of the canonical ring.

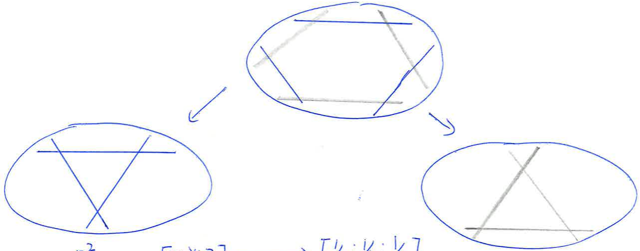

$\mathbb{P}^1_{\mathbb{C}}$
$g=0$
$K(\mathbb{P}^1_{\mathbb{C}})=-1$
Rational curve
Curve of general type
$g\geq 2$ $K=1$
Elliptic curve
$g=1$ $K=0$

Birational equivalence

- One would like to prove similar results in higher dimensions.
- If $\dim X \geq 2$, the birational equivalence relation is non-trivial.
- In dimension 2, any two smooth birational surfaces become isomorphic after finitely many blow ups of smooth points (Zariski, 1931).
- In dimension ≥ 3 the situation is much more complicated, however it is known by work of Wlodarczyk (1999) and Abramovich-Karu-Matsuki-Wlodarczyk (2002), that the birational equivalence relation amongst smooth varieties is generated by blow ups along smooth centers.
- It is easy to see that if X, X' are birational smooth varieties, then $\pi_1(X) \cong \pi_1(X')$ and $R(\omega_X) \cong R(\omega_{X'})$.
- However, typically, they have different Betti numbers eg. $b_2(X) \neq b_2(X')$.



Two birational varieties are connected by a sequence of blow ups along smooth subvarieties.



$\mathbb{P}^2$ $\quad [x:y:z] \longrightarrow [1/x : 1/y : 1/z] \quad$ $\mathbb{P}^2$

Canonical models of surfaces

- It is then natural to try to classify varieties up to birational equivalence.
- We would like to identify a unique "best" representative in each equivalence class: the **canonical model**.
- In dimension 2, the canonical model is obtained by first contracting all -1 curves ($E \cong \mathbb{P}^1$, $c_1(\omega_X) \cdot E = -1$) to get $X \to X_{\mathrm{min}}$,
- Then we contract all 0-curves ($E \cong \mathbb{P}^1$, $c_1(\omega_X) \cdot E = 0$) to get $X_{\mathrm{min}} \to X_{\mathrm{can}}$.
- Note that X_{can} may have some mild singularities (duVal/RDP/canonical). In particular $\omega_{X_{\mathrm{can}}}$ is a line bundle.
- Bombieri's Theorem says that if $\kappa(X) = 2$ (i.e. $\omega_{X_{\mathrm{can}}}$ is ample), then ϕ_5 embeds X_{can} in $\mathbb{P}^N = \mathbb{P}H^0(\omega_X^{\otimes 5})$.
- It follows easily that $X_{\mathrm{can}} \cong \mathrm{Proj} R(\omega_X)$, and $\omega_{X_{\mathrm{can}}} = \mathcal{O}_{X_{\mathrm{can}}}(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_{X_{\mathrm{can}}}$.
- There are some nice consequences.

- As mentioned above, ϕ_5 embeds X_{can} in $\mathbb{P}^N = \mathbb{P}H^0(\omega_X^{\otimes 5})$ as a variety of degree $25c_1(\omega_{X_{\mathrm{can}}})^2$.
- So for any fixed integer $v = c_1(\omega_{X_{\mathrm{can}}})^2$, canonical surfaces depend on finitely many algebraic parameters.
- The number

$$v = c_1(\omega_{X_{\mathrm{can}}})^2 = \lim \frac{\dim H^0(\omega_X^{\otimes m})}{m^2/2},$$

is the **canonical volume**.
- Generalizing this picture to higher dimensions is a hard problem which was solved in dimension 3 in the 80's by work of Mori, Kawamata, Kollár, Reid, Shokurov and others.
- In higher dimension there has been much recent progress which I will now discuss.

Finite generation

Theorem (Birkar, Cascini, Hacon, M^{c}Kernan, Siu 2010)

Let X be a smooth complex projective variety, then the canonical ring $R(\omega_X) = \oplus_{m\geq 0} H^0(\omega_X^{\otimes m})$ is finitely generated.

Corollary (Birkar, Cascini, Hacon, M^{c}Kernan)

If X is of general type ($\kappa(X) = \dim X$), then X has a canonical model X_{can} and a minimal model $X_{\min}$.

Conjecture

- *If $\kappa(X) < 0$, then X is birational to a Mori fiber space $X' \to Z$ where the fibers are Fano varieties ($\omega_F^\vee$ is ample).*
- *If $0 \leq \kappa(X) < \dim X$, then X is birational to a ω-trivial fibration $X' \to Z$ ($\omega_F^{\otimes m} = \mathcal{O}_F$ some $m > 0$).*

Canonical models

- Assume that X is of general type ($\kappa(X) = \dim X$).
- The **canonical model** $X_{\mathrm{can}} := \mathrm{Proj}(R(\omega_X))$ is a distinguished "canonical" (unique) representative of the birational equivalence class of X which is defined by the generators and relations in the finitely generated ring $R(\omega_X)$.
- X_{can} may be singular, but its singularities are mild (canonical). In particular they are cohomologically insignificant (rational sings) so that e.g. $H^i(\mathcal{O}_X) \cong H^i(\mathcal{O}_{X_{\mathrm{can}}})$ for $0 \le i \le \dim X$.
- The "canonical line bundle" is now a $\mathbb{Q}$-line bundle which means that $\omega_{X_{\mathrm{can}}}^{\otimes n}$ is a line bundle for some $n > 0$.
- $\omega_{X_{\mathrm{can}}}$ is ample so that $\omega_{X_{\mathrm{can}}}^{\otimes m} = \phi_m^* \mathcal{O}_{\mathbb{P}^N}(1)$ for some $m > 0$.

Varieties of general type

- In higher dimensions, we define the canonical volume

$$vol(X) = c_1(\omega_{X_{\rm can}})^d = \lim \frac{\dim H^0(\omega_X^{\otimes m})}{m^d/d!},$$

Theorem (Hacon-McKernan, Takayama, Tsuji)

Let V_d be the set of canonical volumes of smooth projective d-dimensional varieties. Then V_d is discrete. In particular $v_d := \min V_d > 0$.

Theorem (Hacon-McKernan, Takayama, Tsuji)

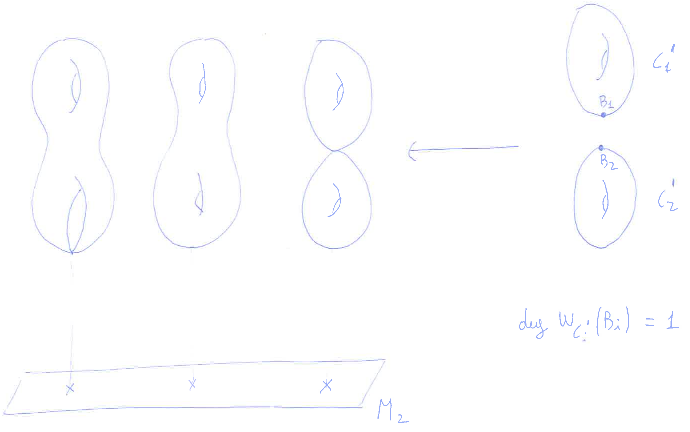
Fix $d \in \mathbb{N}$ and $v \in V_d$, then the set $\mathcal{C}_{d,v}$ of d-dimensional canonical models $X_{\rm can}$ such that $vol(X_{\rm can}) = v$ is bounded (depends algebraically on finitely many parameters, and in particular has finitely many topological types).

Varieties of general type

- The proof relies on first showing that there exists an integer m_d depending on d such that for any $m \geq m_d$, if X is a smooth complex projective variety of dimension d, then $\phi_m : X \dashrightarrow \mathbb{P}^N$ is birational for $m \geq m_d$.
- For fixed volume v, we then obtain an algebraic family $\mathcal{X} \to T$ such that for any X as above with $\mathrm{vol}(\omega_X) = v$, there exists $t \in T$ and a birational isomorphism $X \dashrightarrow \mathcal{X}_t$.
- We then replace $\mathcal{X} \to T$ by a resolution and consider the corresponding relative canonical model.
- There is no known value for v_d, m_d when $d \geq 4$. $m_d = 3, 5, \leq 77$, $v_d = 2, 1, \leq 1/420$.
- Effective results in dimension 3 where obtained by Jungkai Chen and Meng Chen using Reid's Riemann-Roch formula.

Stable curves

- $\mathcal{C}_{d,v}$ can also be compactified by adding stable varieties.
- When $d = 1$ and $v = 2g - 2 > 0$, then $M_g = \mathcal{C}_{1,2g-2}$.
- In order to compactify this space $M_g \subset \bar{M}_g$, we must allow Curves (Riemann surfaces) to degenerate to the well known **stable curves** (Deligne and Mumford 1969).
- A stable curve $C = \cup C_i$ is a union of curves whose only singularities are nodes and ω_C is ample.
- If $\nu_i : C_i' \to C_i \subset C$ denotes the desingularization and B_i is the inverse image of the nodes, then $\nu_i^* \omega_C = \omega_{C_i'}(B_i)$ is ample (we allow logarithmic poles along B_i).
- In higher dimensions there is a similar theory of KSBA moduli spaces (Kollár, Shepherd-Barron, and Alexeev).


C_1'
B_1
B_2
C_2'
deg $W_{C_i'}(B_i) = 1$
M_2

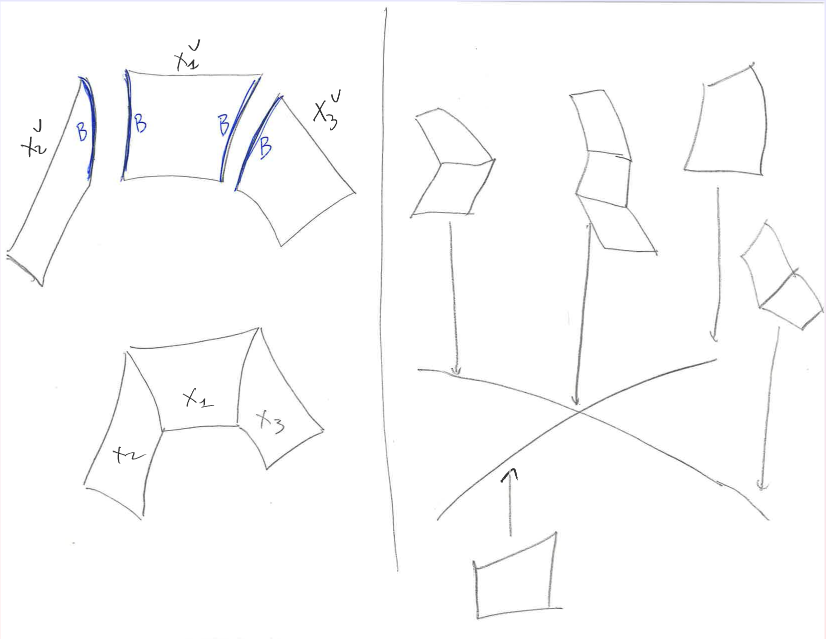
Semi-log-canonical models

- We say that $X = \cup X_i$ is a **slc model** if X is S_2, X_i intersects X_j transversely in codimension 1, ω_X is ample $\mathbb{Q}$-Cartier and if $\nu : \coprod X_i^\nu \to X$ is the normalization, then $\nu_i^* \omega_X = \omega_{X_i^\nu}(B_i)$ where (X_i^ν, B_i) is log-canonical (e.g. X is smooth and B_i has simple normal crossings support).
- We denote $\mathcal{SLC}_{d,v}$ the set of d-dimensional slc models of volume d.

Theorem (Alexeev, Hacon-Xu, Hacon-McKernan-Xu, Kollár, Fujino, Kovács-Patakfalvi, ...)

Fix $d \in \mathbb{N}$ and $v > 0$. Then $\mathcal{SLC}_{d,v}$ is projective.

- Note however that these moduli spaces can be arbitrarily singular (Vakil).
- Moreover $\mathcal{C}_{d,v}$ is not dense in $\mathcal{SLC}_{d,v}$.


$X_1^\vee$
$X_2^\vee$
$X_3^\vee$
B
B
B
B
X_1
X_2
X_3

Boundedness of SLC models

Theorem (Hacon-McKernan-Xu)

Fix $d \in \mathbb{N}$. The set of volumes of d-dimensional slc models is well ordered (satisfies the DCC so that there are no accumulation points from above and in particular there is a positive minimum).

- This generalizes a celebrated result of Alexeev in dimension 2.
- Note that we have accumulation points from below. Eg consider $\mathbb{P}^2$ and B the union of 4 lines. If we do $f : X \to \mathbb{P}^2$ a weighted blow up with weights $(1, n)$ at the intersection of 2 lines then $(X, f_*^{-1}B)$ has volume $1 - \frac{1}{n}$.
- Consider $S = \{vol(K_X + B) | slc\ model,\ \dim X = 2\} \cap [0, M]$.
- S' the set of accumulation points of S, $S^{(n)} = (S^{(n-1)})'$. Is $S^{(k)} = 0$ for $k \gg 0$?

Theorem (Hacon-McKernan-Xu)

Fix $d \in \mathbb{N}$ and $I \subset [0,1]$ a well ordered set. The set of volumes of d-dimensional klt pairs (X,B) where the coefficients of B are in I is well ordered (satisfies the DCC so that there are no accumulation points from above and in particular there is a positive minimum).

- Eg. if $d=1$ and $I=\{1-\frac{1}{n}|n\in\mathbb{N}\}$, then $\mathrm{vol}(K_X+B)=2g-2+\sum b_i$ where $B=\sum b_iB_i$.
- An easy case by case analysis shows that the smallest positive volume is $1/42$ ($g\geq 2$ implies $\mathrm{vol}\geq 2$, $g=1$ implies $\mathrm{vol}\geq 1-1/2$, $g=0$.....).
- As a consequence one can show that if X is a curve of genus $g\geq 2$, then $|\mathrm{Aut}(X)|\leq 84(g-1)$.

- In fact, let $f : X \to Y = X/\mathrm{Aut}(X)$, then $\omega_X = f^*\omega_Y(\sum(1-\frac{1}{r_i})P_i)$ where f is ramified to order r_i at P_i.
- $y = x^r$, $dy = rx^{r-1}dx = ry^{\frac{r-1}{r}}dx$.
- Then $2g-2 = \deg \omega_X = \deg(f) \cdot \deg(\omega_Y(\sum(1-\frac{1}{r_i})P_i) \geq |\mathrm{Aut}(X)| \cdot \frac{1}{42}$.
- In higher dimesion, this says that $\mathrm{vol}(\omega_X) \geq |\mathrm{Aut}(X)| \cdot \mathrm{vol}(\omega_Y(\sum(1-\frac{1}{r_i})P_i)$.
- If v_0 is the minimum of positive volumes of the form $\mathrm{vol}(\omega_Y(\sum(1-\frac{1}{r_i})P_i))$, then $|\mathrm{Aut}(X)| \leq \frac{1}{v_0} \cdot \mathrm{vol}(\omega_X)$.

Intermediate Kodaira dimension.

- Consider now the case $0 \leq \kappa(X) < \dim X$.
- $X \dashrightarrow Z := \mathrm{Proj} R(K_X)$ has positive dimensional general fibers F with $\kappa(F) = 0$.
- Conjecturally F has a minimal model $F \dashrightarrow F'$ such that $K_{F'} \equiv 0$. (True if $\dim X \leq 3$.)
- Typical examples are Abelian Varieties, Hyperkahler varieties and Calabi-Yau's.
- We view these varieties ($K_{F'} \equiv 0$) as the building blocks of varieties of intermediate Kodaira dimension.
- We hope to understand X in terms of the geometry of F' and of its moduli space.
- Unluckily it is not even known if in dimension 3, F' can have finitely many topological types!

Mori Fiber Spaces

- Next we consider varieties with $\kappa(X) < 0$.
- Conjecturally these are the uniruled varieties (i.e. covered by rational curves). This is known if $\dim X \leq 3$.

Theorem (Birkar-Cascini-Hacon-M^{c}Kernan)

Let X be a uniruled variety. Then there is a finite sequence of flips and divisorial contractions $X \dashrightarrow X'$ and a morphism $f : X' \to Z$ such that: $\dim X' > \dim Z$, $\rho(X'/Z) = 1$, $c_1(\omega_{X'}) \cdot C < 0$ for any curve C contained in a fiber of f.

- $f : X' \to Z$ is a **Mori fiber space**.
- The fibers F of f are **Fano varieties** with terminal singularities so that ω_F^{-1} is an ample $\mathbb{Q}$-line bundle.

Mori Fiber Spaces

- Fano varieties are well understood.
- For example $\pi_1(F) = 0$ and for any divisor D, the corresponding ring $R(D) = \oplus_{m\geq 0} H^0(mD)$ is finitely generated.
- We think of Fano varieties as the building blocks for uniruled varieties.
- They play an important role in algebraic geometry and many related subjects.

Fano

- The most important question related to Fano varieties is: **Are Fano varieties with mild singularities** (terminal or even ϵ-log-terminal singularities) **bounded?**
- Several versions of this questions have appeared prominently in the litterature and are known to have many important consequences (existence of Kahler-Einstein metrics, applications to Cremona groups, ...)
- In dimension 2, varieties with terminal singularities are smooth and it is known that there are 10 possibilities (algebraic families).
- In dimension 3, there are 105 families of smooth Fano's (Iskovskih 1989 and Mori-Mukai 1991) and many more families of terminal 3-folds.
- The boundedness of smooth Fano varieties in any dimension was shown by Campana, Nadel, Kollár, Mori and Miyaoka in the early 1990's.

BAB conjecture

- The boundedness of terminal Fano 3-folds was shown by Kawamata (1992) (Kollár, Mori, Miyaoka and Takagi, 2000 for the canonical case).
- The boundedness of (ϵ-log-) terminal toric Fanos was shown by A. Borisov and L. Borisov in 1993 and for ϵ-log terminal surfaces by Alexeev in 1994.
- The **BAB conjecture** claims that for any $\epsilon > 0$ Fano varieties with with ϵ-log terminal singularities are bounded.
- In recent spectacular progress, Caucher Birkar was able to prove that this conjecture is true.

Theorem (Birkar)

The BAB conjecture holds, in particular the set of all terminal Fano varieties in any fixed dimension is bounded.

Characteristic $p > 0$ and mixed characteristic

- Since all of the proofs rely on involved applications of Kodaira Vanishing, they do not work in $\mathrm{char}(p) > 0$.
- Most of what I discussed so far is known in positive characteristic and dimension ≤ 2 with some exceptions:
- Does semistable reduction hold in characteristic $p > 0$? (OK if you fix $\mathrm{vol}(\omega_X)$ and let $p \gg 0$.)
- Does inversion of adjunction work in characteristic $p > 0$ or mixed characteristic?
- The most important/natural question is: Is $R(\omega_X)$ is finitely generated? (OK if $d \leq 2$ or in most cases if $d = 3$, $p > 5$.)
- If X is smooth over a DVR, then is $P_m(X_k) = P_m(X_K)$ for m sufficiently divisible?
- Fix $d > 0$, then for $p \gg 0$, if X is log terminal, then is it CM? (OK if $d = 2$.)