Tarski Lectures: Compact spaces, definability, and measures in model theory

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Berkeley, April 6th, 8th, and 10th, 2009

Introduction to the lecture series I

The themes of the lectures include:

- (i) What (locally) compact spaces can be intrinsically attached to a first order theory T as invariants of T? (There are also interesting invariants of a descriptive set theoretic nature which I will not go into in these lectures.)
- (ii) What interesting mathematical objects (manifolds, Lie groups,..) can be recovered from first order theories?
- (iii) To what extent can the theory of (Haar) measure and integration be lifted from the category of (locally) compact spaces to the category of definable sets in a given theory T?
- Implicit in the above is how we want to view a first order theory T as a mathematical object.

Introduction to the lecture series II

- Although specific theories such as RCF and more general o-minimal theories will play a prominent role in these talks, my concern here is with "pure" model theory, or model theory in and for itself, rather than "applied" model theory.
- In the background is current research on generalizing and adapting the vast machinery of stability theory to unstable theories, as well as work on "continuous model theory".
- There is an interesting resonance with some of the themes and concerns in the earlier days of model theory, represented say in the Proceedings of the 1963 Symposium at Berkeley, which are in a sense being revisited in the light of the "long march" through stability theory.

Introduction to the lecture series III

- I will have to assume familiarity with at least the elementary parts of first order logic and model theory.
- T denotes a consistent first order theory in a language L, i.e. a set of L-sentences closed under logical implication and with a model. Usually T is assumed to be complete: for every L-sentence σ either $\sigma \in T$ or $\neg\sigma \in T$.
- $M, M', N, ..$ will denote L-structures (M for "model".)
- I assume familiarity with notions such as: $M \models \sigma$, and for $\phi(x_1, .., x_n)$ an L-formula with free variables $x_1, .., x_n$ and $\bar{a}$ an n tuple of elements of M, $M \models \phi(\bar{a})$. Also with $M \prec N$ (M is an elementary substructure of N).

Type spaces I

- We start by looking at question (i). You might think that a possible answer is: Let $T = RCF = Th(\mathbb{R}, +, \cdot, <)$. Then $\mathbb{R}$ is the unique locally compact model of T (with topology induced by the ordering).
- But in fact it is a bad answer. In what sense can this standard model, and its topology, be recovered from the "object" T, or characterized purely model-theoretically? It is not prime, or saturated. We'll come back to this.
- Of course there are some compact spaces intrinsically attached to an arbitrary theory T. Namely the *type spaces*, although these on the face of it belong to the syntactic rather than semantic aspect of T.

Type spaces II

- For each n let $F_n(T)$ be the Lindenbaum algebra of T, namely the Boolean algebra of L-formulas $\phi(x_1,..,x_n)$ (in free variables $x_1,..,x_n$)), up to equivalence modulo T.
- The space $S_n(T)$ is the Stone space of $F_n(T)$, namely the set of ultrafilters on $F_n(T)$, or complete n-types of T. The basic open sets of $S_n(T)$ are of the form $\{p(\bar{x}) : \phi(\bar{x}) \in p\}$, for $\phi(\bar{x})$ a formula. As such $S_n(T)$ is totally disconnected, and possibly not so interesting from the point of view of geometry.
- The theory T can actually be presented as a "type-space functor", namely the functor which takes a natural number n to $S_n(T)$. (What are the morphisms?)
- Beware: $S_2(T)$ is NOT $S_1(T) \times S_1(T)$, even as a set. It is because of this that model theory is not reduced to topology.
- This is the same phenomenon as with the Zariski topology on an algebraic variety X. The Zariski topology on $X \times X$ is not the product of the Zariski topology on X with itself.

Type spaces III

- Likewise, if $M \models T$ and $A \subseteq M$, we have $S_n(Th((M,a)_{a \in A})$ which we often write as $S_n(A)$ and call the set of complete n-types over A. (But remember this depends on $Th((M,a)_{a \in A})$.)
- So $S_n(A)$ is the Stone space of the natural Boolean algebra $F_n(A)$.
- If $M \models T$ and $A \subseteq M$ and $b \in M^n$, we have $tp(b/A) \in S_n(A)$. Moreover for any $p \in S_n(A)$ there is an elementary extension of M in which p is "realized".

Semantics: category of definable sets I

- Naively, a definable set is $\phi(M) = \{a \in M^n : M \models \phi(a)\}$, for some model M of T, and formula $\phi(x_1, .., x_n)$ of L.
- It is more reasonable to define a definable set as a functor F_ϕ from $Mod(T)$ (with elementary embeddings as morphisms) to $Sets$ determined by some formula $\phi(x_1, .., x_n)$: namely $F_\phi(M) = \phi(M)$.
- As such the collection of definable sets (of n-tuples) identifies with $F_n(T)$.
- Sometimes we write a definable set as X or X_ϕ and talk about $X(M)$ for a model M.
- Likewise we can speak of sets definable with parameters, or A-definable sets, or sets defined over A.

Semantics: category of definable sets II

- Can a definable set be viewed naturally as a compact space?
- Well consider the formula $0 \leq x \leq 1$ in RCF, then in the model $\mathbb{R}$ it defines the unit interval $[0, 1](\mathbb{R})$ (a compact space). But this does not count, as remarked earlier.
- If $\phi(M)$ is finite for some $M \models T$ then $\phi(M)$ has the same finite size for all M, and the functor F_ϕ has constant value a fixed finite set, which of course IS a compact space (with discrete topology).
- But once $\phi(M)$ is infinite for some model M then by the compactness theorem, $|\phi(N)|$ is unbounded, as N varies, and there is no sense in which the formula, definable set, or functor can be viewed as a compact object.
- Similarly for "type-definable" or "$\wedge$-definable" sets. A type-definable set is given by a collection $\Phi(x_1, .., x_n)$ of formulas, where for a model M, $\Phi(M) = \{a \in M^n : M \models \phi(\bar{x}) \text{ for all } \phi \in \Phi\}$.

Semantics: category of definable sets III

- Then if $\Phi(M)$ is infinite for some model M then by compactness $|\Phi(N)|$ is unbounded as N varies over models of T.
- And if $\Phi(M)$ is finite in all models M of T then Φ is equivalent to a single formula ϕ with F_ϕ constant valued.
- We can slightly enlarge our notion of definability by considering quotients by definable equivalence relations. That is let X be a definable set (even type-definable set), and E a definable equivalence relation on X (meaning what?). Then $(X/E)(M) =_{def} X(M)/E(M)$ for any model M.
- Then EITHER $|(X/E)(M)|$ is unbounded as M varies, OR $(X/E)(M)$ has constant value which is a finite set.
- So far the only definable sets which have a chance of being considered compact sets are the finite ones. But a slight twist will produce something new.

Hyperdefinable sets I

- Let X be a definable (or even type-definable) set and let now E be a *type-definable* equivalence relation on X. (Where X, E could be defined with parameters from some model.)
- For any model M over which X, E are defined, define $(X/E)(M)$ to be $X(M)/E(M)$. We call (the functor) X/E a hyperdefinable set.

Example 1.1

The type space $S_n(T)$ "is" a hyperdefinable set (defined with no parameters).

Hyperdefinable sets II

- For example consider the case $n = 1$. Let $E(x, y)$ be the type-definable equivalence relation given by $\{\phi(x) \leftrightarrow \phi(y) : \phi(x) \in L\}$, and let X be defined by $x = x$. Then as long as all 1-types are realized in M we have a tautological bijection between $S_1(T)$ and $(X/E)(M)$.
- So in fact X/E is "eventually constant", that is if N is an elementary extension of M and all 1-types are realized in M then $(X/E)(M) = (X/E)(N)$.

Hyperdefinable sets III

Definition 1.2
Let X/E be a hyperdefinable set. Call X/E bounded if it is eventually constant. Namely there is a model M_0 (over which X, E are defined) such that whenever $M_0 \prec M$ then $(X/E)(M) = (X/E)(M_0)$. Equivalently, for $\bar{M}$ a sufficiently saturated model, $|(X/E)(\bar{M})| < |\bar{M}|$.

- So through Example 1.1 we have examples of bounded hyperdefinable sets which are not "finite sets".
- In fact Example 1.1 is "universal" as we will shortly explain.

Hyperdefinable sets IV

- It is convenient at this point to replace the category $Mod(T)$ by a fixed very saturated model $\bar{M}$ of T (whose existence may depend on set theory). So for some inaccessible cardinal $\bar{\kappa} > |T|$, $\bar{M}$ is $\bar{\kappa}$ saturated and of cardinality $\bar{\kappa}$.
- $\bar{M}$ is a kind of "proper class" or universe. "Small" or "bounded" means of cardinality $< \kappa$. $M, N, ..$ denote small elementary substructures of $\bar{M}$, and $A, B, ..$ small subsets of $\bar{M}$. Partial types $\Phi(\bar{x})$ are meant to be over small sets of parameters.
- Identify a definable, or type-definable, or even hyperdefinable, set X with $X(\bar{M})$.
- Then it is a fact/theorem that a hyperdefinable set X is bounded just if $X(\bar{M})$ is bounded, i.e. of cardinality $< \bar{\kappa}$.
- This may offend certain sensibilities, but everything I say will have an equivalent syntactic presentation.

Hyperdefinable sets V

Theorem 1.3

Let X/E be a bounded hyperdefinable set, with X, E defined over a model M_0. Let $\pi : X \to X/E$ be the canonical projection. Then
(i) Define $C \subseteq X/E$ to be closed if and only if $\pi^{-1}(C) \subseteq X \subseteq \bar{M}^n$ is type-definable (with parameters from $\bar{M}$). Then this defines a topology (the logic topology) on X/E which is compact (Hausdorff).
(ii) For $b \in X$, b/E depends only on $tp(b/M_0)$, hence π factors through the relevant type space $S_X(M_0)$ (space of complete n-types over M_0 extending "$x \in X$").
(iii) X/E with the logic topology is a continuous image of $S_X(M_0)$.

Hyperdefinable sets VI

- Step I. Proof of (i). Interesting application of the compactness theorem Left as an exercise.
- Step II. Proof of (ii). The proof is the same as the proof that over a model, types coincide with "Lascar strong types". First show that if b, c begin an indiscernible sequence over M_0 then $E(b, c)$. Then show, using "coheir sequences" that if $tp(b/M_0) = tp(c/M_0)$ then there is some infinite I such that both b, I and c, I are indiscernible. It follows from the previous sentence that $E(b, c)$.
- Step III. Now if Z is a type definable set which is $Aut(\bar{M}/M_0)$-invariant then in fact Z is type-definable with parameters from M_0. Hence by (ii), $C \subseteq X/E$ is closed in the logic topology iff $\pi^{-1}(C)$ is type-definable with parameters from M_0.

Hyperdefinable sets VII

- Step IV. Proof of (iii). Let $S_X(M_0)$ be the space of complete types over M_0 extending the partial type "$x \in X$" (a closed subspace of $S_n(M_0)$). By (ii), the map $\pi : X \to X/E$ factors through a surjective map $\rho : S_X(M_0) \to X/E$. Note that the subsets of X which are type-definable over M_0 correspond to the closed subsets of the space $S_X(M_0)$. So we can restate Step II as: $C \subseteq X/E$ is closed if and only if $\rho^{-1}(C)$ is a closed subset of $S_X(M_0)$. This proves (iii).
- So note that the bounded hyperdefinable sets equipped with the logic topology are precisely the continuous images of closed subspaces of Stone spaces over models.

Standard part maps I

- We have seen in Lecture I that bounded hyperdefinable sets are, as compact spaces, continuous images of type spaces.
- Are there interesting, in particular non totally disconnected, spaces arising this way?
- Let M_0 be a structure whose underlying universe is a compact Hausdorff space X, and such that for some basis $\mathcal{U}$ of X, every $U \in \mathcal{U}$ is definable in M_0, by a formula $\phi_U(x)$ say. Let $T = Th(M_0)$.
- Let $E(x, y)$ be the type-definable equivalence relation $\{\phi_U(x) \leftrightarrow \phi_U(y) : U \in \mathcal{U}\}$.
- Then it is a not too hard theorem that (i) E is bounded, and (ii) for M a saturated model of T, M/E with the *logic topology* is *homeomorphic* to X, where moreover the homeomorphism is induced by the standard part map $st : M \to M_0$ ($st :^* X \to X$).

Standard part maps II

- In certain "tame" situations, the equivalence relation E is canonical.
- Consider again RCF, and let I be the unit interval $0 \leq x \leq 1$ considered as a definable set in RCF.
- The equivalence relation E above is precisely $\{|x - y| < 1/n : n = 1, 2, 3, ..\}$, and as above the standard part map witnesses that I/E is homeomorphic to the real unit interval $I(\mathbb{R})$.
- After identifying 0 and 1, $(I, +(mod 1))$ is a group. And it turns out that E is the *finest* bounded type-definable equivalence relation on I which is invariant under this group operation.
- I/E then has a group structure, and with the logic topology, is none other than the circle group S_1.

Standard part maps III

- So modulo the choice of the group structure, we have, from the first order theory RCF, recovered the real unit interval as a topological space (or the real Lie group S_1), without ever imposing from outside any topologies.
- The result on the previous slide generalizes to the unit cube I^n, and in fact to any "definably compact" group G in RCF which is defined with parameters from $\mathbb{R}$:
- Namely the equivalence relation E on G of being "infinitesimally close" is the finest bounded type-definable equivalence relation on G which is invariant under the group operation, and G/E with the logic topology identifies with the compact Lie group $G(\mathbb{R})$ via the standard part map.

Standard part maps IV

- Here and for the rest of the lectures, I tend to concentrate on definable groups and invariant equivalence relations as results have clean statements.
- In tame contexts such as RCF and also $Th(\mathbb{Q}_p)$, the standard part map $st : G \to G(\mathbb{R})$ has an additional model-theoretic feature, DOMINATION, which will figure again in Lecture III.

Lemma 2.1

(G definable in RCF over $\mathbb{R}$ and "definably compact", and $\pi : G \to G(\mathbb{R})$ the standard part map) For any definable subset X of G (with parameters anywhere), for almost all $c \in G(\mathbb{R})$ in the sense of Haar measure, $\pi^{-1}(c)$ is either contained in X or disjoint from X.

G^{00} I

- The results in the previous section had a tautological aspect, in so far as we were recovering, admittedly by purely logical means, objects that we already knew existed.
- We would like to generalize to situations where there is NO extrinsic standard part map. So for example where $G = A(R)$, A an abelian variety defined with parameters from a nonArchimedean real closed field R, and with trivial $\mathbb{R}$-trace.
- We fix for now an arbitrary complete theory T, and a definable group G (as in Lecture I identified with its set of points in a saturated model $\bar{M}$).
- If A is a set of parameters over which G is defined, let G^0_A be the intersection (conjunction) of all A-definable subgroups of G of finite index. Let G^{00}_A be the smallest type-definable over A subgroup of bounded index.

G^{00} II

- Then $G^{00}_A \subseteq G^0_A$ and both are normal type-definable subgroups of G bounded index.
- If G^0_A does not depend on A, we just call it G^0, the definably connected component of G. Likewise for G^{00}, the type-definably connected component of G.
- G/G^{00} with the logic topology is a compact (Hausdorff) topological group, and G/G^0 is its maximal profinite quotient.
- For T ω-stable (countably many types over any countable model), $G^0 = G^{00}$ is definable and of finite index in G.
- For T stable (at most λ^ω many types over any model of size λ), $G^0 = G^{00}$ but may be an infinite intersection of finite index subgroups.
- For T without the independence property (for example o-minimal T), G^{00} exists but may be strictly contained in G^0

G^{00} III

- For T o-minimal (such as RCF), G^0 is definable of finite index, but as we have seen G^{00} may not equal G^0.
- By a Lie group we mean a real analytic manifold with real analytic group structure. When we say that such and such is a compact Lie group we mean it is the underlying topological group of a compact Lie group.

Theorem 2.2

Let G be a "definably compact", definably connected, definable group in $\bar{M} \models T$, where T is o-minimal. Then G/G^{00} with the logic topology is a compact Lie group with dimension equal to the o-minimal dimension of G.

G^{00} IV

- This result (positive solution to the so-called Pillay conjecture), proved in 2005, was the culmination of work by many model-theorists. The crucial case is where G is commutative, and Keisler measures, discussed in Lecture III, were central to the proof.
- The o-minimal dimension of a definable set in an o-minimal structure is a purely model-theoretic "rank" and can for example be defined in a similar way to Morley rank in strongly minimal theories, using that algebraic closure is a "pregeometry".
- The dimension of a Lie group is its dimension as a real manifold, and is well-defined for the underlying topological group.
- We view the canonical surjective homomorphism $\pi : G \to G/G^{00}$ as an *intrinsic* standard part map, or even better as an intrinsic *reduction map*.

Reduction maps I

- Even though there may be no extrinsic standard part map in general, there does exist, in the context of algebraic geometry and in the presence of a valued field, an algebraic-geometric reduction map:
- Let K be a field, $\mathcal{V}$ a valuation subring, $\mathcal{M}$ its maximal ideal, $k = \mathcal{V}/\mathcal{M}$ the residue field, and $\pi : \mathcal{V} \to k$ the canonical ring homomorphism.
- If X is an algebraic variety defined by equations with coefficients from $\mathcal{V}$, then we can apply π to the coefficients to find an algebraic variety $\overline{X}$ defined by equations over k.
- If moreover X is a projective variety, then we also obtain a map $\pi : X(K) \to \overline{X}(k)$.

Reduction maps II

- Take now the case where K is a non-Archimedean (even saturated) real closed field, $\mathcal{V}$ is the "finite" elements, $\mathcal{M}$ the infinitesimals, and then $k = \mathbb{R}$.
- If the definable group G discussed earlier is defined by equations over $\mathcal{V}$, then we have both the intrinsic reduction map $G \to G/G^{00}$ and an algebraic-geometric reduction map. How do they compare? Can one distinguish various behaviours of algebraic-geometric reduction model-theoretically?
- We will complete this lecture with a brief description of the situation for elliptic curves (Davide Penazzi).
- An elliptic curve E over K is something defined by an equation $y^2 = f(x)$ (together with a point at infinity), where f is a cubic over K with distinct roots. E has a group structure given by rational functions, and we take G to be $E(K)$ (in fact to be precise $E(K)^0$ the semialgebraic connected component of $E(K)$), a definable group in K.

Reduction maps III

I will state a proposition then explain the words.

Theorem 2.3

E has good reduction or nonsplit multiplicative reduction if and only in in the structure obtained from $(K, +, \cdot)$ by adding a predicate for G^{00}, a field is definable in G/G^{00} (or more technically G/G^{00} is nonmodular). In these cases the model-theoretic reduction map $G \to G/G^{00}$ "coincides" with the algebraic-geometric reduction map $E(K) \to \overline{E}^{ns}(k)$

- Some explanations:
- E has "good reduction" if the algebraic variety $\overline{E}$ defined over $k = \mathbb{R}$ is nonsingular (essentially if $\overline{f}$ still has distinct roots), in which case $\overline{E}$ is still an elliptic curve, and $\pi : E(K) \to \overline{E}(\mathbb{R})$ is a surjective homomorphism.

Reduction maps IV

- If E does not have good reduction, then $\overline{E}^{ns}$ is nevertheless an algebraic group over $\mathbb{R}$, isomorphic over $\mathbb{R}$ to the multiplicative group (split multiplicative reduction), to SO_2 (nonsplit multiplicative reduction), or to the additive group (additive reduction).
- If we define $E_0(K)$ as the preimage of $\overline{E}^{ns}$ under π, π induces a surjective homomorphism $E_0(K) \to \overline{E}^{ns}(\mathbb{R})$.

Motivation: stability and Haar measure I

- There are various reasons for wanting to introduce or consider measures in the context of model theory or first order definability (e.g. for its own sake, or to obtain applications to analysis).
- My own motivation here is model-theoretic/geometric.
- Let X be an irreducible algebraic variety, defined over a field k and identified with its set of points $X(K)$ in an algebraically closed field K of infinite transcendence degree over k. So X is a definable set, in the structure $(K, +, \cdot)$, defined over k.
- Then it is a fact that there is a unique $\{0, 1\}$-valued, finitely additive measure μ on definable subsets of X, such that $\mu(X) = 1$, μ is $Aut(K/k)$-invariant, and $\mu(Y) = 0$ for any proper subvariety Y of X defined over k.

Motivation: stability and Haar measure II

- In fact this "uniqueness" property extends to and is even characteristic of *stable* theories:
- Let T be a stable theory, and as earlier $\bar{M}$ a saturated model, M_0 a small elementary submodel, and $p(x)$ a complete type over M_0.
- Then there is a unique $\{0,1\}$-valued finitely additive measure on definable sets in $\bar{M}$ (defined by formulas $\phi(x)$ over $\bar{M}$) which extends the type $p(x)$ and is $Aut(\bar{M}/M_0)$-invariant. Explanation!! Forking!! (This property can be taken as the definition of a stable theory.)
- For definable groups there is a cleaner statement. Suppose G is a definable group in $\bar{M}$ such that $G = G^0$. Then there is a unique $\{0,1\}$-valued finitely additive measure on definable subsets of G such that $\mu(G) = 1$ and μ is translation invariant.

Motivation: stability and Haar measure III

- To extend such uniqueness behaviour outside the context of stable theories, requires replacing $\{0,1\}$-valued measures by $[0,1]$-valued measures.
- I will give a rough guide to how this can be accomplished in suitable situations, obtaining results that are new even in the semialgebraic (RCF) context. As earlier we will concentrate on definable groups rather than types.
- Of course in the category of compact groups G we have Haar measure (G-invariant Borel probability measure) which is unique. So in a sense we view existence and uniqueness of Haar measure as analogous to existence and uniqueness of invariant types for connected stable groups, and seek a common generalization.

Keisler measures and NIP I

- A global Keisler measure μ_x is a finitely additive probability measure on definable sets (defined by formulas $\phi(x, ...)$ with parameters from $\bar{M}$). If μ_x is only defined on sets definable over a given set A of parameters, we say μ is a Keisler measure over A.
- When μ is $\{0,1\}$-valued, rather than $[0,1]$-valued, it is the same thing as a complete x-type.
- A Keisler measure μ_x over A is the "same thing" as a regular Borel probability measure on the Stone space $S_x(A)$ of complete x-types over A.
- If μ_x, λ_y are global Keisler measures, one would like to define $(\mu_x \otimes \lambda_y)(\phi(x,y))$ as $\int \mu_x(\phi(x,y)) d\lambda_y$.

Keisler measures and NIP II

- For this "product" to make sense the function $f(b)$ taking $b \to \mu_x(\phi(x,b))$ should be λ_y-measurable.
- T is said to have NIP if for any formula $\phi(x,y)$ there is n_ϕ such that there is no set $\{b_1,..,b_n\}$ in $\bar{M}$ such that for every subset s of $\{1,..,n\}$ there is a_s such that $M \models \phi(a_s,b_i)$ iff $i \in s$. (Any stable theory has NIP, as do unstable theories such as RCF, $Th(\mathbb{Q}_p)$,...)
- A global Keisler measure μ_x is (automorphism) invariant if for some small set A of parameters, μ_x is $Aut(\bar{M}/A)$-invariant.

Lemma 3.1

If T has NIP and μ_x is invariant, then μ_x is Borel definable over some small set A, hence $\mu_x \otimes \lambda_y$ can be defined as above.

Keisler measures and NIP III

- Another ingredient is the Vapnik-Chervonenkis theorem or uniform law of large numbers from probability theory (and learning theory).
- The NIP assumption is precisely the assumption on a family of "events" under which the VC-theorem applies.
- Applying the VC-theorem to type-spaces yields.

Lemma 3.2

Assume T has NIP and μ_x is a Keisler measure over M_0. Fix a formula $\phi(x, y)$, and $\epsilon > 0$. Then there are $p_1(x), .., p_k(x) \in S_x(M_0)$ such that for any $b \in M_0$, $\mu_x(\phi(x, b))$ is within ϵ of the proportion of i such that $\phi(x, b) \in p_i$.

Invariant measures on definable groups I

- Let us fix a theory T with NIP, and a definable group G.
- We will call a definable subset X of G left generic if finitely many left translates of X cover G.
- We will say that G has the "finitely satisfiable generics" (fsg) property if (i) Left and right generics coincide, (ii) the family of nongeneric definable subsets of G forms an ideal (in the Boolean algebra of definable subsets of G), and (iii), there is a small subset A of G such that every generic definable subset of G meets A.
- Examples include ANY definable group in a stable theory, as well as "definably compact" groups in RCF such as $A(K)$, A an abelian variety over $K \models RCF$.

Invariant measures on definable groups II

Theorem 3.3

Let T have NIP and G be a definable group with fsg. Then there is a unique left G-invariant global Keisler measure on G, which is also the unique right G-invariant Keisler measure on G.

- We discuss ingredients of the proof.
- From the fsg assumptions plus the existence of G^{00} we find some left invariant Keisler measure μ on G which is "generic", in the sense that any definable set with positive μ measure is generic. So μ is "finitely satisfiable" in a small set A.
- If λ is another left invariant Keisler measure on G we show that $\mu = \lambda^{-1}$:
- λ can be assumed to be (automorphism) invariant in a strong way, namely definable.

Invariant measures on definable groups III

- The key part of the proof is then to use Lemma 3.2 to obtain that $\mu \otimes \lambda = \lambda \otimes \mu$ (Fubini).
- For X a definable subset of G we compute the measure of $\{(x, y) \in G \times G : x \in yX\}$ in two ways to obtain that $\mu(X) = \lambda(X^{-1})$.
- This suffices.
- These uniqueness statements have appropriate versions for types in NIP theories.

Domination, additional remarks I

- When G is a group definable in a real closed field K, defined over $\mathbb{R}$, then in fact Theorem 3.3 can be recovered from the "domination" statement of Lemma 2.1 (for which no proof was given): for any definable subset Y of G, almost all (in the sense of Haar measure on $G(\mathbb{R})$) fibres of $st : G \to G(\mathbb{R})$, are contained in Y or disjoint from Y.
- Namely this statement, together with uniqueness of Haar measure on $G(\mathbb{R})$ implies a unique invariant Keisler measure on G.
- There is a rather far-reaching but elementary generalization of Lemma 2.1 to arbitrary Borel probability measures on real or p-adic semialgebraic sets, which I now state.

Domination, additional remarks II

Lemma 3.4

Let $X \subseteq \mathbb{R}^n$ be bounded, semialgebraic and let μ be a Borel (so countably additive) probability measure on the space X. If K is a saturated elementary extension of $(\mathbb{R}, +, \cdot)$ then $X(K)$ is piecewise dominated by X and μ. In particular, μ has a unique extension to a Keisler measure μ' on $X(K)$. Likewise with $\mathbb{Q}_p$ in place of $\mathbb{R}$.

One deduces from Lemma 3.4, the "approximate definability" of arbitrary finite valued Borel measures on real and p-adic semialgebraic sets X: for each formula $\phi(x, y)$, closed subset C of $[0, 1]$ and $\epsilon > 0$, there is a formula $\psi(y)$ such that for any $b \in \mathbb{R}$, if $\mu(\phi(x, b)) \in C$ then $\models \psi(b)$ and if $\models \psi(b)$ then $\mu(\phi(x, b))$ is within an ϵ-neighbourhood of C.

Domination, additional remarks III

- Of course Theorem 3.3 also applies to arbitrary "definably compact" groups in o-minimal structures, whether or not they are defined over $\mathbb{R}$.
- In fact in this more general situation, we also have the stronger compact domination result.

Theorem 3.5

G is dominated by the compact Lie group G/G^{00}, equipped with its Haar measure.

References I

A few references for some of the things discussed in these lectures include

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References II

- Many other mathematicians have contributed directly or indirectly to the ideas and results in these lectures. These include:
- I. Ben Yaacov, A. Berarducci, M. Edmundo, H.J. Keisler, A. Onshuus, M. Otero, D. Penazzi, S. Shelah, S. Starchenko.